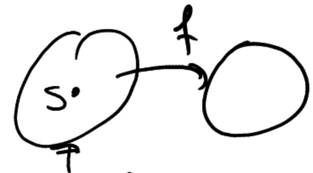
Math 321

Q's 2.3 #3 is f a function



set of all bitstrings.

$\{\lambda, 0, 1, 00, 01, 10, 11, \ldots\}$

3a) $f(s)$ = position of a 0 in S

ex $f(00) = 1, 2$

$00 \to 1$, $00 \to 2$

$11 \to \cancel{\varepsilon}$

$01 \to 1$

$1001 \to 2$, $1001 \to 3$

3b) $f(s)$ = how many 1-bits

$00 \to 0$ $\quad$ $010 \to 1$

$\lambda \to 0$

$1111 \to 4$

Cardinality : Cardinal Numbers

$$\underbrace{0, 1, 2, 3, \ldots, n, \ldots}_{\text{non-neg integers (finite)}} \; ; \; \underbrace{\boxed{\begin{matrix}|\mathbb{Z}^+| \\ \shortparallel \\ \aleph_0\end{matrix}} < \aleph_1 < \aleph_2 < \cdots}_{\text{transfinite cardinal numbers}}$$

this is infinite

$\boxed{\text{th}^{\text{m}}}$ $\mathbb{Q}$ is countable

$\boxed{\text{pf}}$ Goal: find a function from $\mathbb{Z}^+$ to $\mathbb{Q}$ that is a bijection.

consider the set of all $\frac{a}{b}$, a, b are ints and $b \neq 0$ in the form of the table...

f

$1 \to 0$
$2 \to 1$
$3 \to -1$
$4 \to 1/2$
$5 \to -1/2$
$6 \to 2$
$7 \to -2$
$8 \to 1/3$
$9 \to -1/3$
(skip $\pm 2/2$)
$10 \to 3$
$11 \to -3$
$\vdots$

$n=1$ $\quad$ $n=2$ $\quad$ $n=3$ $\quad$ $n=4$ $\quad$ $n=5$

$$\begin{array}{ccccc}
0 & & & & \\
 & \pm\frac{1}{1} & \pm\frac{2}{1} & \pm\frac{3}{1} & \pm\frac{4}{1} \cdots \\
 & \pm\frac{1}{2} & \pm\frac{2}{2} & \pm\frac{3}{2} & \pm\frac{4}{2} \cdots \\
 & \pm\frac{1}{3} & \pm\frac{2}{3} & \pm\frac{3}{3} & \pm\frac{4}{3} \cdots \\
 & \pm\frac{1}{4} & \pm\frac{2}{4} & \pm\frac{3}{4} & \pm\frac{4}{4} \cdots \\
 & \vdots & \vdots & \vdots & \vdots \ddots
\end{array}$$

→ this table contains $\mathbb{Q}$.

Consider the diagonals such that $a+b=n$ and the first diagonal is just holding 0.

any $\frac{a}{b}$ will have a diagonal, that is always of finite length, that it is on.

our bijection is to go along diagonals $1, 2, 3, 4, \ldots$ and count all the $\pm\frac{a}{b}$ that are rational (skip any $\frac{a}{b}$ with common factors)

(b/c f is a bijection)

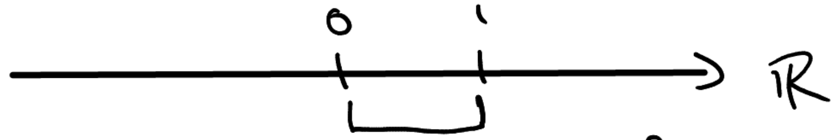
$\to |\mathbb{Q}| = \aleph_0$ (it is countable)

$\boxed{\text{Th}^{\text{m}}}$ $\mathbb{R}$ is uncountable

pf (goal: show there is no bijection from $\mathbb{Z}^+$ to $\mathbb{R}$)

Use contradiction: ($\boxed{\text{goal}}$ show a bijection from $\mathbb{Z}^+$ to $\mathbb{R}$ is always False)

Note:



show all reals from 0 to 1 are uncountable $\to$ $\mathbb{R}$ is uncountable

So we will prove reals from 0 to 1 are uncountable by contradiction.

$\boxed{\text{pf}}$ assume reals from 0 to 1 are countable

Means: bijection from $\mathbb{Z}^+$ to the reals from 0 to 1.

$$\begin{aligned}
1 &\xrightarrow{f} r_1 = 0.d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15} \ldots \\
2 &\to r_2 = 0.d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25} \ldots \\
3 &\to r_3 = 0.d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35} \ldots \\
4 &\to r_4 = 0.d_{41}\, d_{42}\, d_{43}\, d_{44}\, d_{45} \ldots \\
5 &\to r_5 = 0.d_{51}\, d_{52}\, d_{53}\, d_{54}\, d_{55} \ldots \\
&\vdots
\end{aligned}$$

don't like two versions of the same number.

ex) $0.123 = 0.122\overline{9}$

Note: ① $\boxed{\text{All}}$ reals from 0 to 1 are here. $\to$ remove all $\overline{9}$ versions of term. decimals.

② by getting rid of $\overline{9}$ versions we also know each r_i is uniq.

Consider the number $r^* = 0.d_1 d_2 d_3 \ldots$

① Know it is between zero and one.

② make r^*'s d_i to be the following..

book

$$d_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$$

make
$d_1 \neq d_{11}$ and not 0 or 9 $\to$ $r^* \neq r_1$
$d_2 \neq d_{22}$ and not 0 or 9 $\to$ $r^* \neq r_2$
$d_3 \neq d_{33}$ and not 0 or 9 $\to$ $r^* \neq r_3$
$\vdots$

$\to \boxed{r^* \neq r_i \text{ for all } i}$

→ Contradiction $\therefore$ $\mathbb{R}$ are uncountable.

Continuum Hypothesis?

Continuum

$$|\mathbb{R}| = c \neq \aleph_0 \qquad c > \aleph_0$$

$$\boxed{|\mathbb{R}| = \aleph_1}$$

we have shown $|\mathbb{R}| = |P(\mathbb{Z}^+)| = 2^{|\mathbb{Z}^+|} = 2^{\aleph_0} = \aleph_1$