Math 321

Inductive Proofs: show $\forall n\, P(n)$ $n = 1, 2, 3, \ldots$

$\forall n\, P(n) \to$ $P(n)$: "object n has predicate P"

→ $\forall n$: every object n from the U.D.

→ $n = 1, 2, 3, \ldots$: objects have well order

ex) equality proofs.

show $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n = 1, 2, 3, \ldots$

$P(n)$: "$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$"

U.D. : $n=1$, $n=2$, $n=3$, ...

Pf) (1) Basis Step   show $P(1^{st}\text{ case}) \equiv T$?

$P(n=1)$: "$1^3 = \left(\frac{1(1+1)}{2}\right)^2$" $\equiv$ "$1 = 1$" $\equiv T$ ✓

(2) Inductive Step   show $\left[P(k^{th}) \to P(k+1^{st})\right] \equiv T$?

(ex direct proof)   assume $\underbrace{P(k^{th})}_{n=k}$: "$1^3 + 2^3 + \cdots + k^3 \boxed{=} \left(\frac{k(k+1)}{2}\right)^2$"

name this the inductive hypothesis (I.H.)

show $\underbrace{P(k+1^{st})}_{n=k+1}$: "$1^3 + 2^3 + \cdots + (k+1)^3 \overset{?}{\boxed{=}} \left(\frac{(k+1)(k+2)}{2}\right)^2$"

how to show ⊛

tech #1 use true eqn's | tech #2 use expressions

Tech #1

$P(k)$: $1^3+2^3+\dots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$

$1^3+2^3+\dots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2+\frac{(k+1)^3}{1}$

$1^3+\dots+(k+1)^3 = (k+1)^2\left[\frac{k^2}{2^2}+\frac{4(k+1)}{4}\right]$

$1^3+\dots+(k+1)^3 = (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$

$1^3+\dots+(k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$ $P(k+1)$

Tech #2

$1^3+2^3+\dots+(k+1)^3$ ⊛

$= \underbrace{1^3+2^3+\dots+(k)^3}_{\text{by (I.H)}}+(k+1)^3$

$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

$= (k+1)^2\left[\frac{k^2}{4}+(k+1)\right]$

= (by Algebra)

$= \left(\frac{(k+1)(k+2)}{2}\right)^2$ ⊛

(ex) $P(n)$: "$1^2+3^2+5^2+\dots+(2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$"

Cases: $1, 3, 5, 7, \dots, 2n+1$

0 case ($n=0$), 1st ($n=1$), 2nd ($n=2$), 3rd ($n=3$), n^{th} case ($n=n$) ...

Pf

basis: $P(1^{st}\text{ case})$ $\boxed{n=0}$ or 0^{th} odd $=1$: "$1^2 = \frac{(1)(1)(3)}{3}$" $\equiv$ "$1=1$" true ✓

Inductive: show $P(k^{th}) \to P(k+1^{th})$

$n=k$ $\quad$ $n=k+1$

assume $P(n=k)$: " $1^2+3^2+\dots+(2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ " I.H.

show? $P(n=k+1)$; " $1^2+3^2+\dots+(2k+3)^2 \stackrel{?}{=} \frac{(k+2)(2k+3)(2k+5)}{3}$ "

(by expression)

$$1^2+3^2+\dots+(2k+3)^2 = \boxed{1^2+3^2+\dots+(2k+1)^2} + (2k+3)^2$$

by I.H. $$= \boxed{\frac{(k+1)(2k+1)(2k+3)}{3}} + (2k+3)^2$$

$$= \left[\frac{(k+1)(2k+1)}{3} + \frac{(2k+3)}{1}\right](2k+3)$$

$$= \left[\frac{(k+1)(2k+1)+6k+9}{3}\right](2k+3)$$

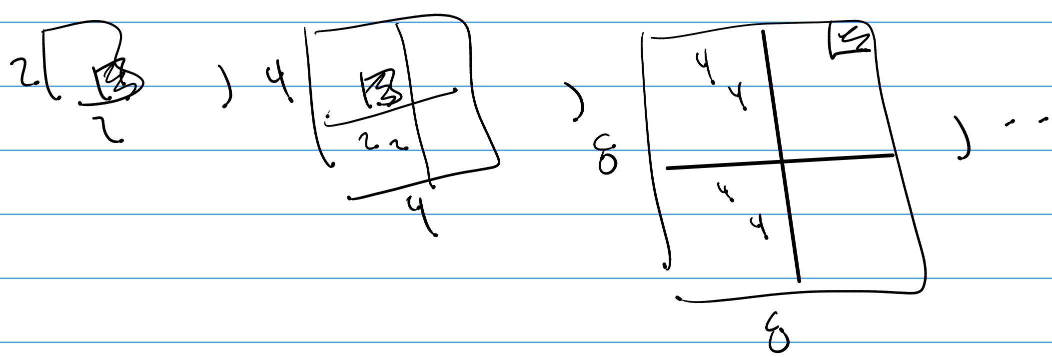
$$= \left(\frac{2k^2+9k+10}{3}\right)(2k+3)$$

$$= \frac{(2k+5)(k+2)(2k+3)}{3}$$

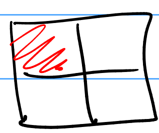
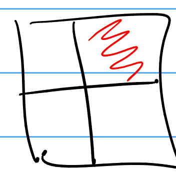
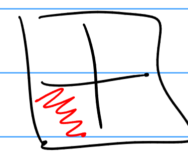
$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$ ✓

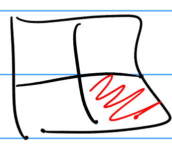
ex tiles  can cover completely any $2^n \times 2^n$ board with one missing piece.

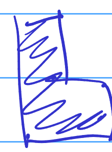
($n = 1, 2, 3, \dots$)

cases 

Basis $P(1^{st}\text{ case})$: tile a 2×2 with a missing piece.

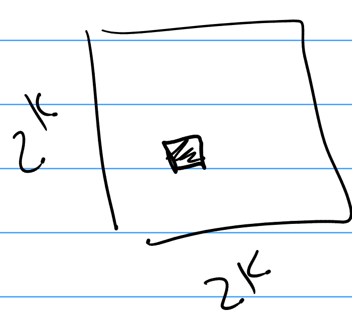
p1)  p2)  p3) 

p4) 

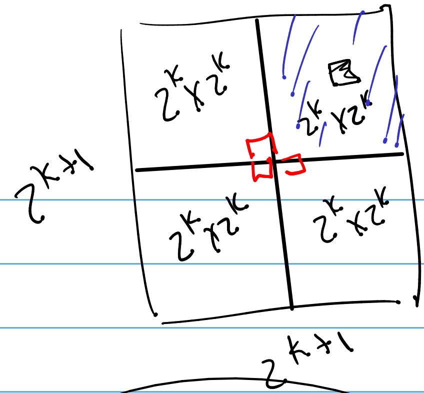
 → tiles all 4.

true

Inductive Assume $P(k^{th}) \rightarrow$ show $P(k+1^{th})$

$P(k^{th}) \rightarrow$  assume it can be tiled.

Show $P(k+1^{st})$



Show it can be tiled,

Attend: write this proof