Math 321

Binomial Thm and Pascal's Triangle

$$(\square+\triangle)^n = \binom{n}{0}\square^n\triangle^0 + \binom{n}{1}\square^{n-1}\triangle^1 + \dots + \binom{n}{j}\square^{n-j}\triangle^j + \dots + \binom{n}{n}\square^0\triangle^n$$

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$n=0$: $\binom{0}{0}$

$n=1$: $\binom{1}{0}$ $\binom{1}{1}$

$n=2$: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$n=3$: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$n=4$: $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$\vdots$

$\frac{4!}{2!\,2!}$ $\frac{4!}{3!\,1!}$

1 $1 = 2^0$

$1 + 1$ $2 = 2^1$

$1 + 2 + 1$ $4 = 2^2$

1 3 3 1 $8 = 2^3$

1 4 $\boxed{6}$ 4 1 $16 = 2^4$

$\vdots$

1 2^0

1 1 2^1

1 2 1 2^2

1 3 3 1 2^3

1, 1, 2, 3, 5, 8, 15,

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

using $(\square + \triangle)^n$

[ex] $\left(x + \frac{2}{x^2}\right)^{100}$

$$= \binom{100}{0}(x)^{100}\left(\frac{2}{x^2}\right)^0 + \binom{100}{1}(x)^{99}\left(\frac{2}{x^2}\right)^{1} + \binom{100}{2}(x)^{98}\left(\frac{2}{x^2}\right)^{2}$$

$$+ \cdots + \binom{100}{j}(x)^{100-j}\left(\frac{2}{x^2}\right)^{j} + \cdots + \binom{100}{100}(x)^{0}\left(\frac{2}{x^2}\right)^{100}$$

[ex] the 13th term

$$\binom{100}{12}(x)^{88}\left(\frac{2}{x^2}\right)^{12} = \frac{100!}{88!\,12!}\, x^{88}\, \frac{2^{12}}{x^{24}} = \boxed{\text{etc}}$$

[ex] give a formula for the x^k

j^{th} $\binom{100}{j} x^{100-j}\left(\frac{2}{x^2}\right)^{j}$

j^{th} $2^j\binom{100}{j} x^{(100-3j)} = ?\ x^k$

let $100-3j = k \rightarrow \frac{100-k}{3} = j$

b/c j is an integer $\frac{100-k}{3} = \text{int}$

$$k^{th} \begin{cases} 2^{\frac{100-k}{3}}\dbinom{100}{\frac{100-k}{3}} x^k & \text{if } 3 \mid 100-k \\ 0 \cdot x^k & \text{if } 3 \nmid 100-k \end{cases}$$

ch 8

8.1 Applications of Recurrence Relations.

(Induction) Basis Step

$\boxed{\text{Inductive Step}}$ $\xrightarrow{\text{formula}}$ Inductive Formula

$\rightarrow$ Recurrence Relation

ex $f_0 = 0 \quad f_1 = 1$ (Basis)

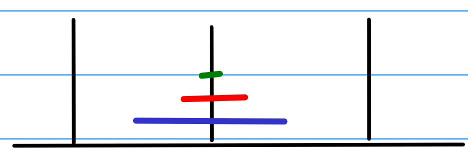
$\boxed{f_n = f_{n-1} + f_{n-2}}$ (Recurrence Relation)

Why study these?

because things are easier to model

using a tech to make new objects from old.

ex Tower



$\boxed{H_1 = 1}$ Basis

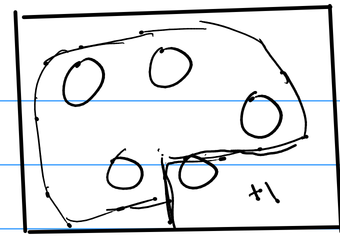
$H_2 = 2H_1 + 1$

$H_3 = 2H_2 + 1$

$\boxed{H_n = 2H_{n-1} + 1}$ Recurrence Relation

Moves: 1, 3, 7, 15, 31, ...

ex S_1, S_2, S_3, S_4, S_5



$$\boxed{a_n = \frac{5}{4} a_{n-1} + 1}$$

$a_0 = A$ but $5 \mid A$ → $a_5 = ?$

Seq $a_0 = A$ $a_1 = \frac{5}{4} A + 1$ $a_2 = \ldots$ $a_5 = ?$

Solve the Rec. Relation

→ Find a closed formula for the relation.

ex $H_1 = 1$, $H_n = 2H_{n-1} + 1$

Seq: $1, 3, 7, 15, 31, \ldots$

how to find a closed formula?

1. guess and check

ex $2, 4, 8, 16, 32, \ldots = \{2^n\}_{n=1,2,\ldots}$

guess $\boxed{H_n = 2^n - 1}$ ✓

check $H_n = 2H_{n-1} + 1$

$(2^n - 1) \overset{?}{=} 2(2^{n-1} - 1) + 1$

$(2^n - 1) \overset{?}{=} 2^n - 2 + 1 = 2^n - 1$ ✓

② Forward or backward iteration

Backward iteration

$$H_n = 2\underbrace{\boxed{H_{n-1}}}_{2H_{n-2}+1} + 1 = 2(2H_{n-2}+1)+1$$

$$H_n = 2^2\underbrace{\boxed{H_{n-2}}}_{2H_{n-3}+1} + 2 + 1 = 2^2(2H_{n-3}+1)+2+1$$

$$H_n = 2^3 H_{n-3} + 2^2 + 2 + 1$$

$\vdots$

$$H_n = 2^{n-1} H_{\underbrace{n-(n-1)}_{1}} + \cdots + 2^2 + 2 + 1$$

$$H_n = 2^{n-1} H_1 + \cdots + 2^2 + 2 + 1$$

$$H_n = \underbrace{2^{n-1} + \cdots + 2^2 + 2 + 1}_{\text{geo sum}} = \frac{2^n - 1}{2-1} = \boxed{2^n - 1}$$

For fun try $\boxed{a_5 = ?}$ try backward iteration.

if $a_0 = A$ that $5 \mid A$

$$a_n = \frac{5}{4} a_{n-1} + 1$$
