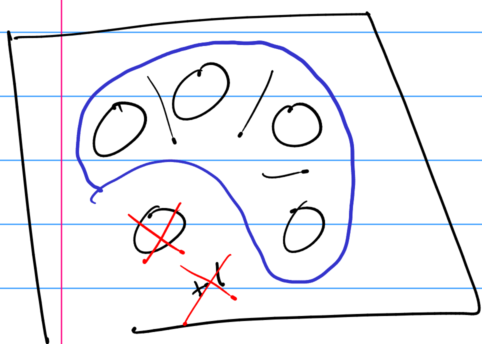
Math 321



$$\boxed{a_n = \frac{5}{4} a_{n-1} + 1}$$ recurrence relation

Using recurrence relations

a_n = expression involving 'older' values $a_{n-1}, a_{n-2}, a_{n-3}, \dots a_{n-k}$

ex bit strings of length n with no 11 in them.

a_n = # of bit strings with no 11

a_n < start with a $\boxed{0}$ $\boxed{\text{can't have 11}}$ a_{n-1} — $n-1$ bits

or

start with a $\boxed{10}$ $\boxed{\text{can't have 11}}$ a_{n-2} — $n-2$

$$a_n = 1 \cdot a_{n-1} + 1 \cdot a_{n-2}$$

Rec. Relation

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

Basis?

$n = 1$ strings: 0, 1

$n = 2$ strings: 00, 01, 10, 11

$a_1 = 2$ $a_2 = 3$

$a_1 = 2 \quad a_2 = 3 \qquad a_n = a_{n-1} + a_{n-2}$

seq: $2, 3, 5, 8, 13, 21, \boxed{34}$

$\quad\;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7$

34 bitstrings of length 7 have no 11 in them.

---

to get a_{1000}, I need $a_1, a_2, \ldots, a_{999}$!

if you have a $\boxed{\text{closed function}}$ $a_n = f(n)$

finding this is $\boxed{\text{Solving}}$ a recurrence relation.

8.2 Solving Recurrence Relations.

(1) guess and check

ex $\quad H_n = 2H_{n-1} + 1 \qquad H_1 = 1$

seq: $1, 3, 7, 15, 31, \ldots$

guess $\quad H_n = 2^n - 1$

(2) Forward / Backward Iteration

ex

$$a_n = \frac{5}{4}\boxed{a_{n-1}} + 1 = \left(\frac{5}{4}\right)^2 \boxed{a_{n-2}} + \left(\frac{5}{4}\right) + 1$$

$$\left[\frac{5}{4}(a_{n-2}) + 1\right] \qquad \left(\frac{5}{4}a_{n-3} + 1\right)$$

$$a_n = \left(\frac{5}{4}\right)^3 a_{n-3} + \left(\frac{5}{4}\right)^2 + \left(\frac{5}{4}\right) + 1$$

$$a_n = \left(\frac{5}{4}\right)^j a_{\underbrace{\boxed{n-j}}_{=0}} + \cdots + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^2 + \left(\frac{5}{4}\right)^1 + \left(\frac{5}{4}\right)^0$$

$$a_n = \left(\frac{5}{4}\right)^n a_0 + \boxed{\left(\frac{5}{4}\right)^{n-1} + \cdots + \left(\frac{5}{4}\right)^2 + \left(\frac{5}{4}\right)^1 + \left(\frac{5}{4}\right)^0}$$

geometric sum $\quad ar^k$

$$\frac{\left(\frac{5}{4}\right)^n - 1}{\frac{5}{4} - 1} = 4\left(\frac{5}{4}\right)^n - 4$$

$$a_n = \left(\frac{5}{4}\right)^n a_0 + 4\left(\frac{5}{4}\right)^n - 4$$

$$\boxed{a_n = (a_0 + 4)\left(\frac{5}{4}\right)^n - 4}$$

Solution

Puzzle: $a_5 = ?$ given $5 \mid a_0$

$$a_5 = (a_0 + 4)\left(\frac{5}{4}\right)^5 - 4 = (a_0 + 4)\left(\frac{3125}{1024}\right) - 4$$

let $a_0 + 4 = 1024$

$\Rightarrow a_0 + 4 = 1024 \quad \therefore \; \underline{a_0 = 1020} \quad \therefore 5 \mid 1020$

$a_5 = 3125 - 4 = \boxed{3121}$

$\rightarrow$ use guess and check like we do in Diff Eq.

Def

$$\boxed{\begin{gathered} a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-\boxed{k}} \\ c_i \in \mathbb{R} \text{ and } c_k \neq 0 \end{gathered}}$$

* linear homogeneous recurrence relation of degree k with constant coeff.

ex

$f_n = f_{n-1} + f_{n-2}$

$a_n = 3a_{n-2} + 7a_{n-5}$

$\cancel{a_n = a_{n-1} \cdot a_{n-2}}$

$\cancel{a_n = \frac{5}{4} a_{n-1} + 1}$

---

ex

$a_n = a_{n-1} + a_{n-2}$

exponential soln

guess $a_{\boxed{n}} = r^{\boxed{n}}$ (r is some const)

check

$a_n = a_{n-1} + a_{n-2}$

$r^n = r^{n-1} + r^{n-2}$

$$r^n = \frac{r^n}{r} + \frac{r^n}{r^2} \rightarrow r^2 \cdot \cancel{r^n} = r \cdot \cancel{r^n} + \cancel{r^n} \quad \boxed{r \neq 0}$$

$$r^2 = r + 1 \rightarrow r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r = \frac{1+\sqrt{5}}{2} \qquad r = \frac{1-\sqrt{5}}{2}$$

Soln's $\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\left(\frac{1-\sqrt{5}}{2}\right)^n$

$$\boxed{a_n = \boxed{c_1}\left(\frac{1+\sqrt{5}}{2}\right)^n + \boxed{c_2}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Soln to $a_n = a_{n-1} + a_{n-2}$

Initial Values $a_0 = 0$ $a_1 = 1$

$$0 = c_1 + c_2 \rightarrow c_2 = -c_1$$

$$1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$2 = c_1(1+\sqrt{5}) - c_1(1-\sqrt{5})$$

$$2 = \sqrt{5}\,c_1 + \sqrt{5}\,c_1 \rightarrow c_1 = \frac{1}{\sqrt{5}}$$

$$c_2 = -\frac{1}{\sqrt{5}}$$

$$f_n = f_{n-1} + f_{n-2} \qquad f_0 = 0 \quad f_1 = 1$$

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

fib. seq