Math 321

Q's/ $H_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j} = \sum_{k=1}^{j} \frac{1}{k}$

Induction: $\forall n P(n)$ n is from $e_1, e_2, e_3, \dots$

Pf) ① (Basis) show $P(e_1)$ is true.

② (Inductive)

a) "weak" assume $\boxed{P(e_k)}$, show $P(e_{k+1})$

b) "strong"

assume $\boxed{P(e_1) \wedge P(e_2) \wedge \dots \wedge P(e_k)}$, show $P(e_{k+1})$

Inductive Hypothesis

Conjecture: $H_{2^n} \leq 1 + n$, $n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$

$P(n)$: "$H_{2^n} \leq 1 + n$"

Scratch

$H_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n}$

$\{2^n\}_{n=0,1,2,\dots} \longrightarrow 1, 2, 4, 8, 16, 32, 64, \dots$

2^0 $H_1 = 1 \leq 1 + 0$ ←

2^1 $H_2 = 1 + \frac{1}{2} \leq 1 + 1$ ←

2^2 $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \leq 1 + 2$ ←

2^3 $H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \leq 1 + 3$ ←

$\vdots$

2^n $H_{2^n} = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} \leq 1 + n$ ←

Conjecture $H_{2^n} \le 1+n$; $n=0,1,2,\ldots$

Note: $H_{2^n} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n}$

Pf by "weak" induction

(i) (Basis) 1st case $n=0$ $\quad H_{2^0} \le 1+0$

$1 \le 1+0$ true

(2) (Inductive)

assume: k^{th} $\quad \boxed{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2^k} \le 1+k}$

show: $k+1^{st}$ $\quad 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k+1}} \overset{?}{\le} 1+k+1$

know: $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k} \le 1+k$

$\rightarrow$ $1+\frac{1}{2}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^k+2^k} \le 1+k+\frac{1}{2^k+1}+\ldots+\frac{1}{2^k+2^k}$

$\rightarrow$ $1+\frac{1}{2}+\ldots+\frac{1}{2^{k+1}} \le 1+k+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^k+2^k}$

$\le 1+k+\frac{1}{2^k}+\frac{1}{2^k}+\ldots+\frac{1}{2^k} = 1+k+2^k\left(\frac{1}{2^k}\right)$

$\rightarrow$ $1+\frac{1}{2}+\ldots+\frac{1}{2^{k+1}} \le 1+k+1$

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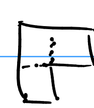
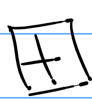
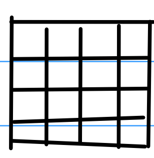
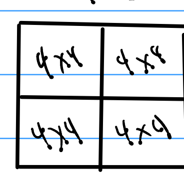
ex Conjecture: any $2^n \times 2^n$ board with one piece missing ($n=1,2,3,\ldots$) can be tiled by  tiles

e_1:  missing one spot

e_2:  Missing one spot

e_3: 8×8 missing one spot



$P(n)$ is yes or no to tile a $\underbrace{2^n \times 2^n}_{\text{missing one spot}}$ with ⌐ tile

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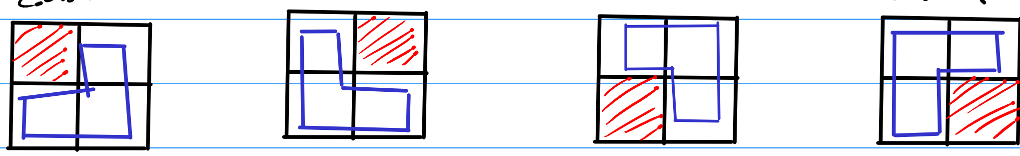
① (Basis) tile a 2×2 with one missing

Case 1 Case 2 Case 3 Case 4



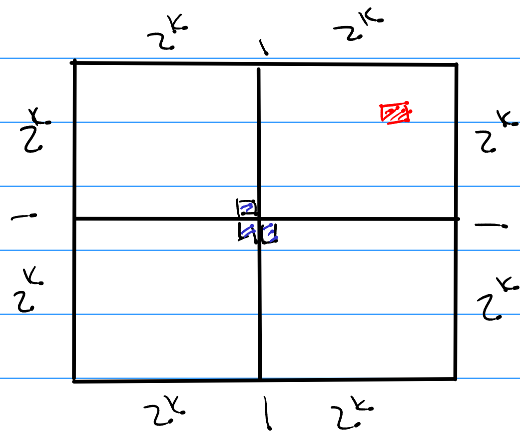
True

② (Inductive Step)

assume: k^{th} tile $2^k \times 2^k$ missing one spot

show: $k+1^{st}$ tile $2^{k+1} \times 2^{k+1}$ missing one spot

$2^{k+1} = 2 \cdot 2^k$



$2^{k+1} \times 2^{k+1}$

$2^{k+1} \times 2^{k+1}$ is four $2^k \times 2^k$. One of these four is missing one spot. The other three are not.

a) by I.H. we can tile the one $2^k \times 2^k$ with a missing spot.

b) What about the other three $2^k \times 2^k$?
remove a tile from shared corner. We can now tile them by I.H. Fill in shared corner with ⌐ tile.

Fund thm of Arith

$n \geq 2$. n is prime or a uniq. prod. of primes in non-dec. order.

to prove this ① Existence $n =$ prod. of primes

② Uniq.