Math 321

Ch 8 Back to <u>inductive / recursive relations</u>

8.1 why?

ex $f_0 = 0$, $f_1 = 1$ ← Basis

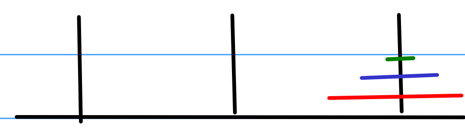
$\boxed{f_n = f_{n-1} + f_{n-2}}$ $n = 2, 3, 4, \ldots$

recursive / inductive relation

(relationship of old to new) → old elements come together to make new elements

→ this is a natural way to model.

ex tower of hanoi



$H_1 = 1$

$H_2 = H_1 + 1 + H_1 = 2H_1 + 1$

$H_3 = H_2 + 1 + H_2 = 2H_2 + 1$

$H_n = 2H_{n-1} + 1$

Model

Basis: $H_1 = 1$

Recursive relation: $H_n = 2H_{n-1} + 1$

1

Seq: 1, 3, 7, 15, 31, 63, ...

$H_1, H_2, H_3, H_4, H_5, H_6, \ldots$

(2) recursive → closed formula | function

Solving the recurrence relation

How: ex $H_n = 2H_{n-1} + 1$, $H_1 = 1$

tech #1 Guess Seq: 1, 3, 7, 15, 31, 63, ...

$$\boxed{a_n = 2^n - 1 \quad n = 1, 2, 3, \ldots}$$

Check:

$H_1 = 1 \qquad a_1 = 2^1 - 1 = 1$

$H_n = 2H_{n-1} + 1 \rightarrow 2^n - 1 \stackrel{?}{=} 2^1(2^{n-1} - 1) + 1$

$2^n - 1 \stackrel{?}{=} 2^n - 2 + 1$

true

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tech #2 iteration (forward / backward)

$H_1 = 1$

$H_n = 2 \cdot H_{n-1} + 1$

$H_1 = 1$

$H_2 = 2 \cdot 1 + 1 = 2 + 1$

$H_3 = 2(2+1) + 1 = 2^2 + 2 + 1$

$H_4 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$

$\vdots$

$H_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0$

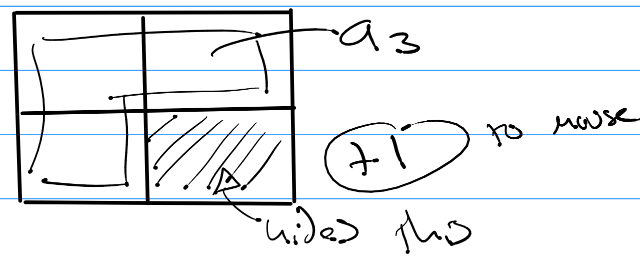
$$\sum_{k=1}^{n} ar^k = a\left(\frac{r^{n+1} - 1}{r - 1}\right)$$

$$H_n = \sum_{k=0}^{n-1} 2^k = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

Modeling with rec. relations

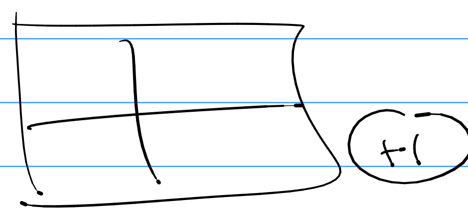
ex 4 kids. Went around town to get all the cookies.
→ take them all home and share them. Wait one day

Over night: Kid #4



$$a_4 = \frac{4}{3} a_3 + 1$$

Kid #3 $a_3 = \frac{4}{3} a_2 + 1$



Kid #2 $a_2 = \frac{4}{3} a_1 + 1$

Kid #1 $a_1 = \frac{4}{3} a_0 + 1$

a_0 is what is there in the morning.

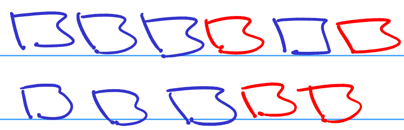
Notice: $4 \mid a_0$

Seq: a_0, a_1, a_2, a_3, a_4 ?

Basis: $4 \mid a_0$

Induction: $a_n = \frac{4}{3} a_{n-1} + 1 \quad n = 1, 2, 3, 4$

(ex) how many tiles of length n used that are ,  never have a  in it?

ex: 

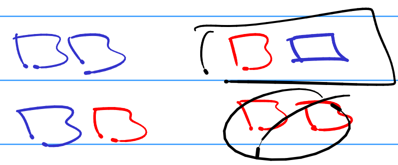
is ok (count)

is not ok (do not count)

C_n = # of ways to tile n length w/o 

Basis: $C_1 = 2$

$C_2 = 3$



Induction: $C_n = (1 \text{ blue tile}) \wedge (\text{rest}) \text{ or } \overset{2\text{-tiles}}{(\text{red}|\text{blue})} \wedge (\text{rest})$

$$C_n = 1 \cdot C_{n-1} + 1 \cdot C_{n-2}$$

$$C_n = C_{n-1} + C_{n-2}$$

Model:

$C_1 = 2$
$C_2 = 3$ basis

recursive $\boxed{C_n = C_{n-1} + C_{n-2}}$

seq: 2, 3, 5, 8, 13, ...

$C_n = f(n)$