Math 511

Systems of linear eqn's

ex
$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ x_1 \quad\quad + x_3 = 2 \\ \quad\quad x_2 - 4x_3 = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

x_1, x_2, x_3 variables

general

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

a_{ij}

row col

as matrices:
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

size: $m \times n$

augmented:
$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Solve: you $\boxed{\text{know}}$ substitution and elimination

Strict triangular system

ex

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ \quad\quad\ \ x_2 + x_3 = 1 \\ \quad\quad\quad\quad\ \ x_3 = 4 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Solve by back substitution

ex

$x_3 = 4$

$$\boxed{(3, -3, 4)}$$

$$\begin{aligned} x_1 + \quad\quad &= 3 \\ x_2 \quad &= -3 \\ x_3 &= 4 \end{aligned}$$

Solve using <u>equivalent system</u>

<u>non strict triangular</u> $\rightarrow$ <u>strict triangular</u>

Elementary Row Operations (do not change soln set)

1. Interchange rows
2. replace row by itself multiplied by a non-zero real.
3. replace row by itself plus a multiple of another row.

ex

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow[r_2, r_3]{\text{swap}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & -1 & 2 & 1 \\ 2 & 1 & 1 & 5 \end{array}\right]$$

$r_3 - 2r_1 = \text{New } R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & -1 & 2 & 1 \\ 0 & -3 & -5 & 13 \end{array}\right] \xrightarrow[-1r_3]{-1r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & 5 & -13 \end{array}\right]$

$r_3 - 3r_2 = \text{New } R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -10/11 \end{array}\right]$

$\left(\frac{1}{11}\right) r_3 = \text{New } R_3$

Solve by back solve

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possible soln's (or non-soln's)

consistent — inconsistent

(end of a problem)

at the start we can call systems one of 3 types

1. <u>determined</u> unknowns = number of eqn's (square coeff matrix)
2. <u>under-determined</u> unknowns > number of eqn's
3. <u>over-determined</u> unknowns < number of eqn's

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det

$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 0 & 4 \end{array}\right]$

<u>under</u>

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \end{array}\right]$

<u>over</u>

$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{array}\right]$

"solve" use elim. row ops to try to get to strict triangular.

3 possible end results.

① one only soln

ex

determined or over det →

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← strict triangular

② no soln

determined over-det under-det →

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

③ ∞ soln's

det under det over det →

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row Echelon Form

① 1st non zero elem of a row is a one.

② non-zero elements of rows are to the right of those above.

ex $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

ex $\left[\begin{array}{cccc|c} 1 & 2 & 4 & 7 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

start a system:

$$\left[\begin{array}{c|c} A & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \text{row echelon form} & \begin{matrix} d_1 \\ \vdots \\ d_n \end{matrix} \end{array}\right]$$

use elem. row ops

called Gaussian elimination.

Note: for tomorrow do Gaussian elimination!