Math 511

reduced row echelon

$$\begin{cases} x + 2y - z = 1 \\ x \quad\; - 2z = 3 \\ 2x - y + z = 4 \\ \quad\; y + 3z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 0 & -2 & 3 \\ 2 & -1 & 1 & 4 \\ 0 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & -2 \\ 0 & -5 & 3 & 2 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & -5 & 3 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 7/18 \\ \hline 0 & 0 & 0 & -37/18 \end{array}\right] \leftarrow \text{no soln}$$

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$$[\quad] \quad [\quad] \xrightarrow{\text{gauss}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & \boxed{1} & 0 & -5 \\ 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

changing problem to make it have a soln

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\text{soln } (13, -5, 2)}$$

ex
$$\begin{bmatrix} & \\ & \end{bmatrix} \xrightarrow[\text{elim}]{\text{gaussian}} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 1 & 3 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -2 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$x_4 = 1$
$x_3 = 1$
$x_2 =$ free (any real number) α
$x_1 = -3 - 2\alpha$

soln $(x_1, x_2, x_3, x_4) = \boxed{(-3-2\alpha, \alpha, 1, 1)}$

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ex
$$\begin{array}{c} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & -3 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

let $x_4 = \alpha$ (any real)

$x_3 = 1 - \alpha$
$x_2 = 1 - \alpha$
$x_1 = -3 - 2\alpha$

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ops for Matrices / Vectors.

know

$A + B$
αA ← scalar mult.
$A - B$

$\vec{u} \cdot \vec{v}$

Name
scalar product
dot product
$\boxed{\text{inner product}}$

Scalar prod $\quad [v_1\ v_2\ v_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = v_1x_1 + v_2x_2 + v_3x_3$

(Matrix) · (col vector)

$$\left\{\begin{array}{l} \boxed{a_{11}x_1 + \cdots + a_{1n}x_n} \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{array}\right. \rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \vec{a}_{1_1} \cdot x \\ \vec{a}_{1_2} \cdot x \\ \vdots \\ \vec{a}_{1_m} \cdot x \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \vec{a}_{1_1} \\ \vec{a}_{1_2} \\ \vdots \\ \vec{a}_{1_m} \end{bmatrix} x$$

$$A x = \begin{bmatrix} \vec{a}_{1_1} x \\ \vec{a}_{1_2} x \\ \vdots \\ \vec{a}_{1_m} x \end{bmatrix}$$

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also

$$\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{array}\right. \rightarrow x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 a_{1_2} + \cdots + x_n a_{1_n}$$

$$A x = \boxed{x_1 a_{1_1} + x_2 a_{1_2} + \cdots + x_n a_{1_n}}$$

$$A = [a_1\ a_2 \cdots a_n] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = [a_1\ a_2 \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

is a linear combo of A's columns

Note: Solve $Ax = b$

find x_i so $x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$

Thm: $Ax = b$ if and only if

b is a linear combination of A's columns.

Note: $$Ax = \begin{bmatrix} \vec{a}_1 x \\ \vec{a}_2 x \\ \vdots \\ \vec{a}_n x \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

(Matrix)(Matrix)

$$\underset{m \times k}{A}\ \underset{k \times n}{B} = A\,[b_1\ b_2 \cdots b_n]$$

$$= [Ab_1\ \ Ab_2 \cdots Ab_n]$$

$$A\mathbf{b}_1 = \begin{bmatrix} \vec{a}_1 \mathbf{b}_1 \\ \vec{a}_2 \mathbf{b}_1 \\ \vdots \\ \vec{a}_m \mathbf{b}_1 \end{bmatrix}$$

So $\underset{m \times k}{A}\ \underset{k \times n}{B} = \underbrace{[\,c_{ij}\,]}_{m \times n}$ where $\boxed{c_{ij} = \vec{a}_i \mathbf{b}_j}$

ex

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}} \underset{3 \times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}} = \begin{bmatrix} 7 & 4 \\ 1 & -2 \end{bmatrix}$$

Opp A^T if $A = [a_{ij}]$

then $A^T = [a_{ji}]$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Note: if $A^T = A \rightarrow$ call A symmetric

1.4 Matrix Algebra — and variables

we also study properties of the operators.

Thm α, β are scalars A, B, C are matrices

① $A+B = B+A$

② $(A+B)+C = A+(B+C)$

③ $(AB)C = A(BC)$

④ $A(B+C) = AB + AC$

⑤ $(A+B)C = (AC + BC)$

⑥ $(\alpha\beta)A = \alpha(\beta A)$

⑦ $\alpha(AB) = (\alpha A)B = A(\alpha B)$

⑧ $(\alpha+\beta)A = \alpha A + \beta A$

⑨ $\alpha(A+B) = \alpha A + \alpha B$

Transpose

$$(A^T)^T = A$$

$$(\alpha A)^T = \alpha A^T$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

b/c $A(BC) = (AB)C$

applied to $A(AA) = (AA)A$

New op $\boxed{A^K = \underbrace{AAA \ldots A}_{K\text{-times}}}$