Math 511

Q's 1.5 12c $AX + B = X$ given A, B, C

$$AX + B + (-1)B = X + (-1)B$$
$$AX + 0 = X + (-1)B$$
$$AX = X + (-1)B$$
$$(-1)X + AX = (-1)X + X + (-1)B$$
$$(-1)X + AX = 0 + (-1)B$$
$$(-1)IX + AX = (-1)B$$
$$((-1)I)X + AX = (-1)B$$
$$\boxed{((-1)I + A)}X = (-1)B$$

$$\underbrace{((-1)I + A)^{-1}((-1)I + A)}_{I}X = ((-1)I + A)^{-1}[(-1)B]$$

$$X = ((-1)I + A)^{-1}[(-1)B]$$

$$X = (-I + A)^{-1}(-B)$$

so $X = \left[\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\right]^{-1} \cdot \begin{bmatrix} -6 & -2 \\ -2 & -4 \end{bmatrix}$

$$X = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -6 & -2 \\ -2 & -4 \end{bmatrix}$$

2

6 later

Elem. row ops vs Elem. Matrices.

system of linear eqns $\longrightarrow$ $\boxed{Ax = b}$

Note! E_{type1} (row swap) E_{type1} is I swap row i, j

$E_{type1}^{-1} = E_{type1}$

ex

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Swap rows 2, 4

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

E_{type2} ($row_i = c \cdot row_i$) $E_{type2} = I$ with c in a_{ii} position

E_{type2}^{-1} is E_{type2} with $1/c$ in a_{ii} spot

E_{type3} ($row_i = row_i + c \cdot row_j$)

E_{type3} = is I with c is in the a_{ij} spot

E_{type3}^{-1} = is E_{type3} with $-c$ in a_{ij} spot

A is 4x4 new row3 by taking $\underline{row3} + (-2)\,\underline{row1}$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

System of Eqn's

$\boxed{\text{Solve!}}$

Matrix Algebra

[augmented matrix]

$\boxed{A}x = b$

gauss-jordan

1st elem. row op $\longrightarrow$ $E_1 A x = E_1 b$

2nd elem row op $\longrightarrow$ $\boxed{E_2 E_1 A} x = E_2 E_1 b$

reduced row ech. [ans?]

$E_k \cdots E_2 E_1 A x = E_k \cdots E_2 E_1 b$

Def: if $B = E_n \cdots E_3 \cdot E_2 \cdot E_1 \cdot A$

call A, B to be $\boxed{\text{row equivalent}}$

If A is $n \times n$ has a uniq soln.

as system of eqn's $[A \mid b] \xrightarrow[\text{Jordan}]{\text{gauss}} [I \mid \text{ans}]$

So $Ax = b$

$$E_1 A x = E_1 b$$

$$E_2 E_1 A x = E_2 E_1 b$$

$$\vdots$$

$$(E_k \cdots E_2 E_1 A) x = E_k \cdots E_2 E_1 b$$

ans

So $(E_k \cdots E_2 E_1) A = I$

So $A^{-1} = (E_k \cdots E_2 E_1)$

Th^m A is n×n. The following are logically equivalent

1. A is non-singular (it's invertible, A^{-1} exists)
2. $A\cdot X = 0$ has only the unique soln $X = 0$
3. A is row equivalent to I

→ use to find A^{-1} by aug. matrices.

Ex $\begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}^{-1}$

$A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$

$E_1 A = [\quad]$

$E_2 E_1 A = [\quad]$

$\underbrace{(E_k \cdots E_2 E_1)}_{} A = I$

Notice: $E_k \cdots E_2 E_1 = E_k \cdots E_2 E_1 I$

$[A \mid I]$

$[E_1 A \mid E_1 I]$

$[E_2 E_1 A \mid E_2 E_1 I]$

$\vdots$

$[I \mid A^{-1}]$

Ex $\begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}^{-1} \longrightarrow \left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$

$row_1 = row_1 + (-1) row_2$ $\qquad \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 3 & 1 & 0 & 1 \end{array}\right]$

$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Note:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$row_2 = row_2 + (-3) row_1$ $\qquad \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & -5 & -3 & 4 \end{array}\right]$

$E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|cc} 1 & 2 & 1 & -1 \\ 0 & 1 & 3/5 & -4/5 \end{array}\right]$

$row_1 = row_1 + (-2) row_2$ $\qquad \left[\begin{array}{cc|cc} 1 & 0 & -1/5 & 3/5 \\ 0 & 1 & 3/5 & -4/5 \end{array}\right]$

$E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$\qquad ||$

$\qquad A^{-1} : I$

$$A^{-1} = E_4 E_3 E_2 E_1$$

Note: $X = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -6 & -2 \\ -2 & -4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & 3 & -6 & -2 \\ 3 & 1 & -2 & -4 \end{array}\right]$$

$$\rightsquigarrow$$

$$\left[I \,\middle|\, A^{-1}\begin{bmatrix} -6 & -2 \\ -2 & -4 \end{bmatrix} \right]$$

Idea: $X^{-1}Y = ?$ $\quad [X \mid Y] \overset{\text{G}}{\rightsquigarrow} [I \mid X^{-1}Y]$

Factorization

$$12 = 1 \cdot 12$$
$$= 2 \cdot 6$$
$$= 3 \cdot 4$$
$$= 2 \cdot 2 \cdot 3$$

Consider: $\{A \mid I\}$

Gauss $\rightsquigarrow$

$$\left[\begin{array}{c|c} \text{upper triangular} & \\ \| & \\ U & \end{array} \right]$$

Note: $\boxed{\text{upper triangular}}$

$$\begin{bmatrix} x & \cdots & x \\ 0 & \ddots & x \end{bmatrix}$$ $a_{ij} = 0$ below diag.

$\boxed{\text{lower triangular}}$

$$\begin{bmatrix} x & \ddots & 0 \\ x & \cdots & x \end{bmatrix}$$ $a_{ij} = 0$ above diag.

$$E_n \cdots E_2 E_1 A = U$$

$$A = (E_1^{-1} E_2^{-1} \cdots E_n^{-1}) U$$

Note: if all your E_i were type 3

$$E_1^{-1} E_2^{-1} \cdots E_n^{-1} = \text{lower triangular} = L$$

So $\boxed{A = L \cdot U}$ LU factorization.