Math 511

LU factorization: use only type 3 elem matrices / ops to make zeros below diagonal.

$$\underbrace{\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 4 & 1 \end{bmatrix}}_{A} \xrightarrow[r_3 + (-\frac{1}{2}) r_1]{r_2 + \frac{1}{2} r_1} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 3 & -1/2 \end{bmatrix} \xrightarrow{r_3 + (-3) r_2} \underbrace{\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -11 \end{bmatrix}}_{U}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U$$

$$A = \underline{E_1^{-1} E_2^{-1} E_3^{-1}} U = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -11 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -11 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -11 \end{bmatrix}}_{U}$$

Q's

$A = LU$

Why?

$$A\mathbf{x} = \mathbf{b}$$

$$LU\mathbf{x} = \mathbf{b}$$

$$U\mathbf{x} = L^{-1}\mathbf{b}$$

1.6 Partitioned Matrices

$$A = \left[\begin{array}{c|c|c} a_{11} & \cdots & a_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline a_{m1} & \cdots & a_{mn} \end{array}\right] = \left[\, a_1 \mid a_2 \mid \cdots \mid a_n \,\right]$$

$$= \left[\begin{array}{c} \vec{a}_1 \\ \hline \vec{a}_2 \\ \hline \vdots \\ \hline \vec{a}_r \end{array}\right]$$

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

partitioned matrix

(partitioned matrix)(partitioned matrix) ← follow the normal rules of matrix mult.

ex)

$$\underset{2 \times 3}{\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right]} \underset{3 \times 1}{\left[\begin{array}{c} B_{11} \\ \hline B_{21} \\ \hline B_{31} \end{array}\right]} = \underset{\substack{2 \times 1 \\ \text{partition}}}{\left[\begin{array}{c} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ \hline A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \end{array}\right]}$$

A^{-1} exists $\rightarrow$ A is non-singular $\rightarrow$ A is invertible

Thm: A^{-1} exists

logically equiv. as $AX = 0$ has only triv. soln $X = 0$

logically equiv as $\boxed{A \text{ is row equiv to } I.}$

$\rightarrow$ use this to find A^{-1} by

$$[A \mid I] \underset{\text{row ops}}{\rightsquigarrow} [I \mid A^{-1}]$$

Does A^{-1} exist?

Ex: $[A \mid I]$

$$\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \;\middle|\; D \right]$$

impossible to make left I

So A^{-1} doesn't exist

2.1 **Idea:** Does some mathematical tool exist to help us determine if A^{-1} exists before we look for it?

only have $[A \mid I] \rightsquigarrow [I \mid A^{-1}]$

1x1 $A = [a]$

$$[a \mid 1] \xrightarrow{1/a\, r_1} [1 \mid 1/a]$$

only when $a \neq 0$

$a = 0$ no inv.

$a \neq 0$ have inv.

2x2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{r_2 + \frac{-c}{a} r_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{a \cdot r_2} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \boxed{ad-bc} & -c & a \end{array}\right]$$

have inv. $a \neq 0$ $ad - bc \neq 0$ do not have inv $ad - bc = 0$

def Determinant of A = $\underline{\det(A)} = |A|$

$\det(A) \neq 0 \rightarrow A^{-1}$ exists

$\det(A) = 0 \rightarrow A^{-1}$ does not exist

$$\det\left(\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\right) = (2) - (2) = 0$$

$$\det([3]) = 3$$

$$\det\left(\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\right) = (3) - (-4) = 7$$

<u>3x3</u>

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[r_3 = r_3 + \frac{-a_{31}}{a_{11}} r_1]{r_2 = r_2 + \frac{-a_{21}}{a_{11}} r_1} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}a_{21} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \frac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix}$$

$$a_{11} \det(\quad) \neq 0$$

do algebra ...

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{31} - a_{13}a_{31}a_{22} \neq 0$$

$$\rightarrow a_{11}\left(\boxed{a_{22}a_{33} - a_{32}a_{23}}\right) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}\left(\boxed{a_{21}a_{32} - a_{31}a_{22}}\right) \neq 0$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Def ① M_{ij} is A with row i, col j removed

Minor of a_{ij}

② Cofactor of a_{ij}

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

Cofactor expansion of $\det(A)$

(along a row or col $\boxed{\text{you}}$ choose)

ex row 1

$$\boxed{\begin{aligned} \det(A) &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\ &\quad + \cdots + (-1)^{1+n} a_{1n} |M_{1n}| \end{aligned}}$$