Math 511

~~Q's?~~

by cofactor expansion (pick row/col)

$$\det(A) = a_{11}|M_{11}| - a_{21}|M_{21}| + a_{31}|M_{31}| + \dots + (-1)^{n+1} a_{n1}|M_{n1}|$$

ex by col 1

pick cofactor expansion along col 2

ex

$$\begin{vmatrix} 1 & 2 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 4 & 0 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & 0 & 4 \\ 3 & 1 & 0 \\ 4 & 1 & 3 \end{vmatrix} + 0$$

$$= -2\left[3\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} + 0\right] - 2\left[1\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 + 4\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}\right]$$

= finish

Cost? — how many mult/divides

5x5

4x4 4x4 4x4 4x4 4x4 — $\boxed{5 \text{ mult}}$

3x3 3x3 3x3 3x3 — 4 mult

$5! = 5 \cdot 4 \cdot 3 \cdot 2$ mult

60

2x2 2x2 2x2 — 2 mult.

2x2

$|5\times 5|$ <u>be about</u> $2 \cdot 5!$ $\sim 5!$

$20\times 20 \sim 20!$

$100\times 100 \sim 100!$ } can't be done computationally!

Idea: $\boxed{(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^{1/5}}$

$e^{\ln(x)} = x$

$\ln(a^b) = b\ln(a)$

$\ln(a\cdot b) = \ln(a) + \ln(b)$

$$\boxed{e^{\frac{\ln(x_1)+\ln(x_2)+\ln(x_3)+\ln(x_4)+\ln(x_5)}{5}}}$$

Properties of $\det(A)$

① $\det(A^T) = \det(A)$

★ ② if A is upper and/or lower triangular

ex $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} = 1\left[2\cdot(-1)\right]$

$\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

(3) if any row (col) is all zeros $\det(A) = 0$

goal: $\det(A)$ vs $\det\begin{pmatrix}\text{matrix that is} \\ \text{row equiv to } A\end{pmatrix}$

$\det(E_{type_1} A) \overset{?}{=}$ (row swap)

$\det(E_{type_2} A) \overset{?}{=}$ ($row_i = c \cdot row_i$)

$\det(E_{type_3} A) = ?$ ($row_i = row_i + c \cdot row_j$)

---

(1) $\det(E_{type_1} \cdot A)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

vs $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = (-1)\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

So $\boxed{\det(E_{type_1} A) = (-1)\det(A)}$

use A: $\underline{\det(E_{type_1})} = \det(E_{type_1} \cdot I)$

$= (-1)\det(I) = \underline{\underline{-1}}$

$$\boxed{\det(E_{type_1} \cdot A) = \det(E_{type_1})\det(A) = (-1)\det(A)}$$

② $\det(\underset{\uparrow}{E_{type2}} \cdot A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \boxed{ca_{i1}} & \cdots & ca_{in} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = c\det(A)$

mult. row i by c

$$\boxed{\det(E_{type2} \cdot A) = c\det(A)}$$

So $\det(E_{type2}) = \det(E_{type2} \cdot I) = c\det(I) = c$

$\therefore$ $$\boxed{\det(E_{type2} \cdot A) = \det(E_{type2})\det(A) = c\det(A)}$$

Lemma $\quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad A_{ij} = (-1)^{i+j}|M_{ij}|$

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & i = j \\ 0 & i \neq j \end{cases}$$

③ $\det(\underset{\uparrow}{E_{type3}} \cdot A) = \begin{vmatrix} a_{11} & \cdots & & a_{1n} \\ \boxed{a_{i1} + ca_{j1}} & \boxed{a_{i2} + ca_{j2}} & \cdots & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix}$

$\text{row}_i = \text{row}_i + c \cdot \text{row}_j$

$\det(\quad) = a_{11}|M_{11}| - a_{21}|M_{21}| + a_{31}|M_{31}| + \cdots + (-1)^{n+1} a_{n1}|M_{n1}|$

$(a_{i1} + ca_{j1})|M_{11}|$

So $\left| E_{type3} A \right| = \det(A) + c \cdot \left(\cancel{\begin{matrix} \text{row } j \\ \text{minors of} \\ \text{row } i \end{matrix}} \right)^{0}$

$$= \det(A)$$

$$\therefore \boxed{\det(E_{type3} A) = \det(A)}$$

So $\det(E_{type3}) = \det(E_{type3} \cdot I) = \det(I) = 1$

$$\boxed{\det(E_{type3} \cdot A) = \det(E_{type3}) \det(A) = 1 \cdot \det(A)}$$

Thm in general $\det(AB) = \det(A)\det(B)$

So $\det(A)$ by cofactors (really costly)

A is row equiv to U by gaussian elimination

$$U = E_k \cdots E_2 E_1 A$$

$$\det(U) = \det(E_k) \cdots \det(E_2)\det(E_1)\boxed{\det(A)}$$

$$\boxed{\det(A) = \frac{\det(U)}{\det(E_k) \cdots \det(E_2)\det(E_1)}}$$ det by elimination

gaussian elimination is $\sim n^3/3$ mult/div.

$(10 \times 10) \to$ cofactor $10!$

$\to$ elimination $\frac{10^3}{3} \approx 300$

2.3 A_{ij} cofactors.

$$\text{adjoint of } A = \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \text{adj } A}$$