Math 511

V is a vector space.

a) set of objects : "vectors"

b) two operations ; $v + w$, αv

given u, v, w are in V α, β are scalars

Axioms   **closure**   (C1) $\alpha v \in V$ (C2) $v + w \in V$

addition: (A1) $u + v = v + u$ (A2) $(u+v)+w = u+(v+w)$

(A3) $\exists\, 0 \in V \ni' u + 0 = u$ (A4) $\exists\, -v \ni' v + (-v) = 0$

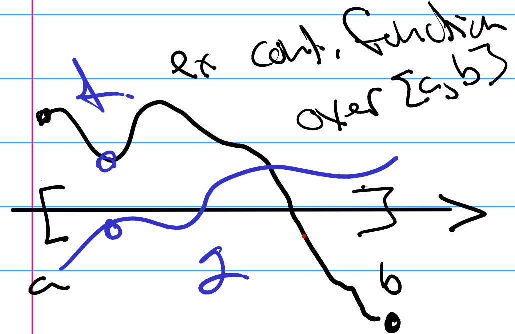
scalar prod (A5) $\alpha(u+v) = \alpha u + \alpha v$ (A6) $(\alpha+\beta)u = \alpha u + \beta u$

(A7) $(\alpha\beta)u = \alpha(\beta u)$ (A8) $1 \cdot u = u$

(ex) $\mathbb{R}^n$ space with normal vector addition, scalar vector mult.

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Unusual   objects : all continuous functions over interval $[a,b]$



ops: ① $(f+g)(x) = f(x) + g(x)$

② $(\alpha f)(x) = \alpha f(x)$

<u>Vector Space?</u> check all 10 axioms

(C2) $(f+g)(x) = \underline{f(x)} + \underline{g(x)}$ still a cont. function? (yes)

(C1) $(\alpha f)(x) = \alpha f(x)$ still a cont. function? (yes)

<u>Addition:</u> (A1) $\underline{f+g} = \underline{g+f}$? yes!

$\underline{(f+g)(x)} = f(x)+g(x) = g(x)+f(x) = \underline{(g+f)(x)}$

Continue for A2 to A8 (show all are true)

So

$$\boxed{\begin{array}{l} \text{Cont. functions over } [a,b] \\ \text{with } (f+g)(x) = f(x)+g(x) \quad \text{defined} \\ \qquad (\alpha f)(x) = \alpha f(x) \\ \text{is a Vector Space} \end{array}}$$

<u>Notation:</u> $C[a,b]$

<u>another:</u> $\boxed{P_n}$ n-term polynomial vector space

<u>objects:</u> $p(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_{n-1}x^{n-1}$

ex P_7 $p(x) = 5 + 4x^3$

$= 5 + 0x + 0x^2 + 4x^3 + 0x^4 + 0x^5 + 0x^6$

or $p(x) = c_{n-1}x^{n-1} + \dots + c_2x^2 + c_1x + c_0$

in P_7: $p(x) = 4x^3 + 5$

$= 0x^6 + 0x^5 + 0x^4 + 4x^3 + 0x^2 + 0x + 5$

Ops: $(p+q)(x) = p(x) + q(x)$

$(\alpha p)(x) = \alpha p(x)$

P_3: $p(x) = 2 - x + 3x^2$ $\quad q(x) = 1 + 2x - 3x^2$

$(p+q)(x) = 3 + x + 0x^2 \in P_3$

Additional properties

thm ① $0\mathbf{v} = \mathbf{0}$

ex P_3 what is $\mathbf{0}$ of P_3?

$\boxed{z(x) = 0 + 0x + 0x^2}$

ex $C[a,b]$ what is $\mathbf{0}$? $\quad z(x) = 0$

② If $\mathbf{x} + \mathbf{y} = \mathbf{0}$ then $\mathbf{y} = -\mathbf{x}$

③ $(-1)\mathbf{v} = -\mathbf{v}$

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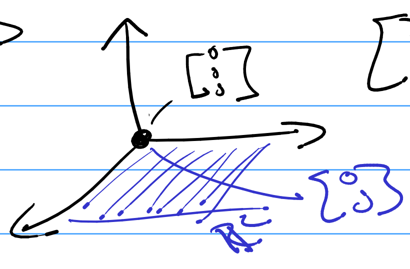
ex $\mathbb{R}^3$



$\begin{bmatrix}x\\y\\0\end{bmatrix} = \mathbb{R}^2$ is a subset of $\mathbb{R}^3$

and $\mathbb{R}^2$ is a vector space

$\rightarrow \mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

Idea: given a set S as a subset of V, a vector space. Goal: is to show S is a vector space as well.

then S is a subspace of V.

Def S is a non-empty subset of V

and

① $\mathbf{0} \in S$ show $\mathbf{0}$ element is in S.

show closure
② $\alpha v \in S$ when $v \in S$
③ $v + u \in S$ when $v, u \in S$

$\rightarrow$ S is a subspace of V.

Subspace?:

① $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_3 = x_1 + x_2 \right\}$

or a diff notation $S = \left\{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \right\}$

S is a subset of $\mathbb{R}^3$

① is $\mathbf{0}$ in S $\quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$ yep!

② $v \in S$ means $\alpha \begin{bmatrix} v_1 \\ v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha(v_1 + v_2) \end{bmatrix}$ + ? yep.

(3) $v + u \in S$?

$$\begin{bmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_1+u_2 \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ (v_1+v_2)+(u_1+u_2) \end{bmatrix}$$ +? yep!

So S is a subspace.

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ex (2) $S = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right\}$

check: (1) is $\vec{0}$ in S? $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$ ($0 \neq 1$)

no it is not a subspace.

Trivial

(1) $S = \{ \vec{0} \}$

(2) $S = V$