Math 511

Q's 3.1 #3 $\mathbb{C}$, complex numbers (objects $(a)+(b)i$) with $i=\sqrt{-1}$

ops $(a+bi)+(c+di)=(a+c)+(b+d)i$

$\alpha(a+bi)=(\alpha a)+(\alpha b)i$

(C1) $\mathbf{v}$ $+\mathbf{u}\in\mathbb{C}$

check: $(a+c)+(b+d)i\in\mathbb{C}$

real + real $\qquad$ real + real

$\boxed{\text{is}}$ real $\qquad$ is real

real + real $\cdot i$ is a complex number (by def)

Subspaces: to show S is a subspace

(subset of V, a vector space, that is also a vector space)

only need to show ① $\mathbf{0}\in S$

if $\mathbf{x},\mathbf{y}$ are in S

② $\mathbf{x}+\mathbf{y}\in S$

③ $\alpha\mathbf{x}\in S$

Trivial ones

① $S=\{\mathbf{0}\}$ $\qquad$ $V=P_5$

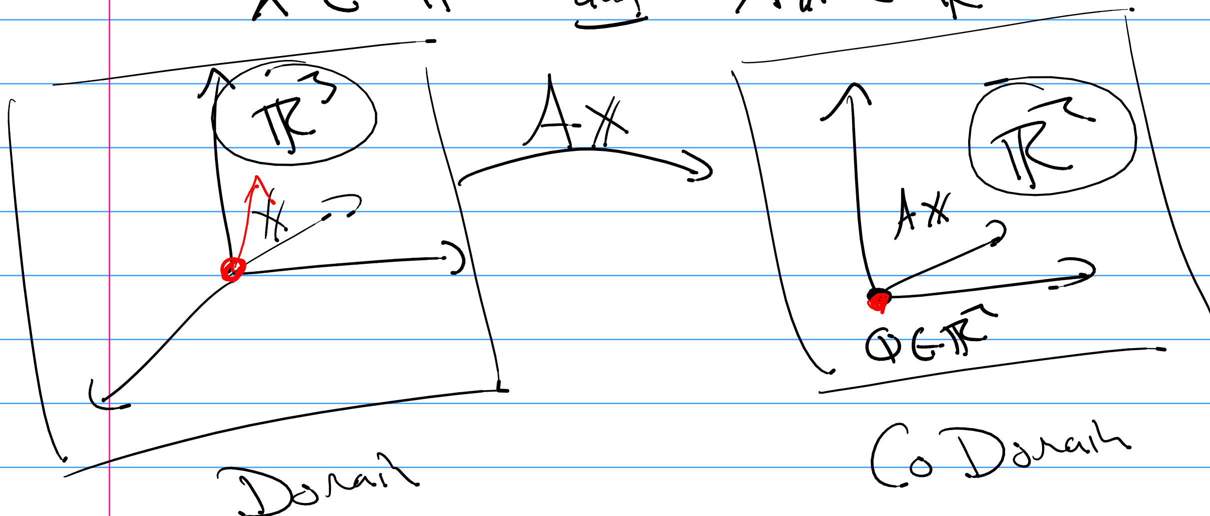
$S=\{0+0x+0x^2+0x^3+0x^4\}$

② $S=V$

A is $m \times n$, then $A\mathbf{x} = \mathbf{0}$

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ $\Rightarrow$ $\underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

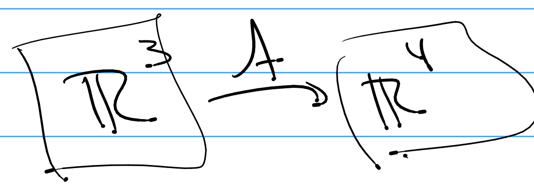
$\mathbf{x} \in \mathbb{R}^3$ and $A\mathbf{x} \in \mathbb{R}^2$



ex $A_{4\times3}$ $\qquad$ $A\mathbf{x} \in \mathbb{R}^4$

$\underbrace{4\times3 \quad 3\times1}_{4\times1}$

$\mathbf{x} \in \mathbb{R}^3$



Solve $A\mathbf{x} = \mathbf{0}$, $A_{m\times n}$

Consider $S = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$

→ S is a subspace of $\mathbb{R}^n$

① $\mathbf{0} \in S$? $\quad \underset{m\times n}{A} \underset{\in \mathbb{R}^n}{\mathbf{0}} = \underset{\in \mathbb{R}^m}{\mathbf{0}}$ $\quad$ yes

② If $\mathbf{x}, \mathbf{y} \in S$ → $\mathbf{x} + \mathbf{y} \in S$

<u>Note:</u> $\mathbf{x} \in S$ means $A\mathbf{x} = \mathbf{0}$

$\mathbf{y} \in S$ mean $A\mathbf{y} = \mathbf{0}$

what about $\mathbf{x}+\mathbf{y} \overset{?}{\in} S$ $\quad A(\mathbf{x}+\mathbf{y})$ $\boxed{\text{Yes}}$

$= A\mathbf{x} + A\mathbf{y}$

$= \mathbf{0} + \mathbf{0} = \mathbf{0}$

(3) $\alpha \mathbf{x} \overset{?}{\in} S$

check $A(\alpha \mathbf{x}) = \alpha(A\mathbf{x}) = \alpha \mathbf{0} = \mathbf{0}$ $\boxed{\text{Yes}}$

Call this set the null-space of A is $m \times n$

$$N(A) = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$$

Ex $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ $\qquad N(A) = ?$

what all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $A\mathbf{x} = \mathbf{0}$

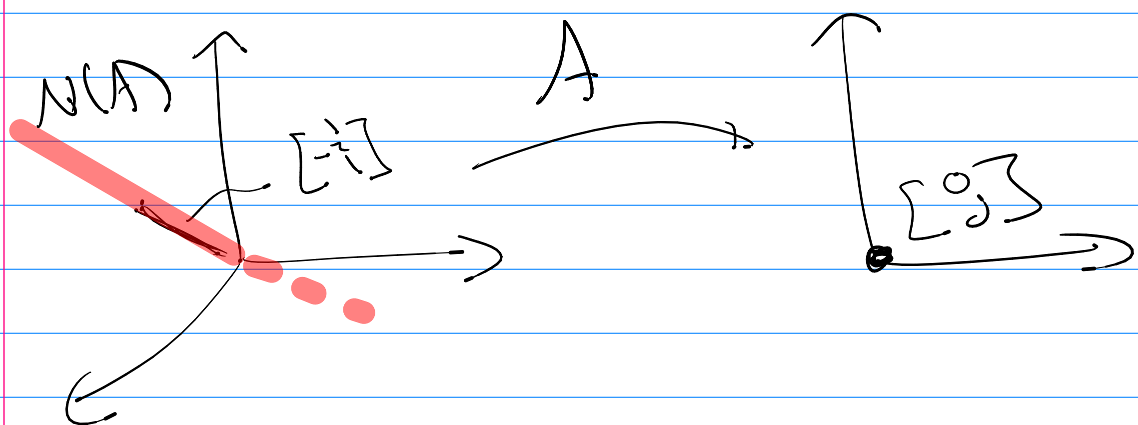
<u>Solve:</u> $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$x_3 = \alpha \to x_2 = -2\alpha \quad x_1 = \alpha$

Sol: $X = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So $N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; \alpha \text{ is any number} \right\}$



Note: find $N(A)$: $[A \mid 0] \rightsquigarrow$ solve (Gauss or Gauss-Jordan)

RREF $[A \mid 0] \longrightarrow$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

free: $x_2 = \alpha \qquad x_4 = \beta$

$x_5 = 0$

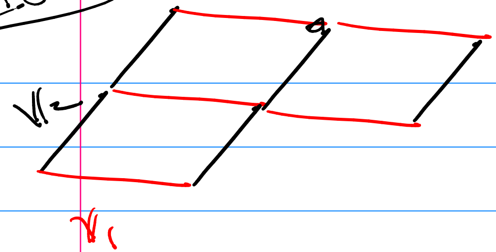
$x_3 = -\beta$

$x_1 = -2\alpha - 3\beta$

$$X = \begin{bmatrix} -2\alpha - 3\beta \\ \alpha + 0\beta \\ 0\alpha - \beta \\ 0\alpha + \beta \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$N(A) = \left\{ \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; \alpha, \beta \text{ are any number} \right\}$$

Idea



$\alpha_1 v_1 + \alpha_2 v_2$ span of v_1, v_2

Def: ① $\{v_1, v_2, \dots, v_k\}$ are vectors in V

② linear combo of them $= \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$

③ the set of all $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$
given α_i are any scalar

is called the span of $\{v_i\}$

$$\text{Span}(v_1, v_2, \dots, v_k)$$

So before

ex $N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle| \ \alpha \text{ is any number} \right\}$

Now: $N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$

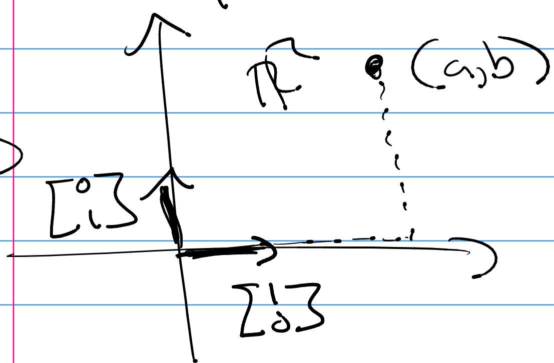
ex
$$N(A) = \left\{ \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \middle| \ \alpha, \beta \text{ are any number} \right\}$$

Now: $N(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$

Th^n Span($v_1, v_2, \ldots, v_k$) is a subspace of V

Q?: is there a set of v_i that allow you to make every v in V?

ex



get to $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

Def: if Span($v_1, v_2, \ldots, v_k$) $= V$

we call $\{v_1, v_2, \ldots, v_k\}$ a <u>spanning set</u>

Find "good" spanning sets

Part 1 How many vectors are really needed?

ex $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ these do span $\mathbb{R}^2$

$$\begin{bmatrix}a\\b\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\end{bmatrix} + \alpha_3\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$$ can be solved for all a,b

Notice: $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$