Math 511

Q's/ $\underline{\mathbb{R}^3}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is $\mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \mathbb{R}^3$?

Solve

is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = ?$

(A) $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ?$

(B) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \text{ans} \\ 0 & 0 & 1 & \end{array}\right]$

(C) $\begin{cases} \alpha_1 + \alpha_2 = x \\ \alpha_2 = y \\ \alpha_1 + \alpha_3 = z \end{cases} \xrightarrow[\text{subs. elim?}]{\text{Solve}} \boxed{\begin{aligned} \alpha_1 &= x - y \\ \alpha_2 &= y \\ \alpha_3 &= z - x + y \end{aligned}}$

So

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha_1 \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} + \alpha_2 \overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + \alpha_3 \overset{v_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\exists$ soln

$\therefore$ spanning set

ex $\begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix} = -6 v_1 + 5 v_2 + 13 v_3$

ex) $\mathrm{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right)$

Show this is a spanning set for $\mathbb{R}^2$

$$\begin{bmatrix}x\\y\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\end{bmatrix} + \alpha_3\begin{bmatrix}3\\1\end{bmatrix}$$

~~Solve~~ $\begin{bmatrix}1 & 1 & 3\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

by Aug. Matrix $\left[\begin{array}{ccc|c}1 & 1 & 3 & x\\0 & 1 & 1 & y\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 2 & x-y\\0 & 1 & 1 & y\end{array}\right]$

$\underline{\alpha_3 \text{ is free}}$ let $\alpha_3 = c$

$\alpha_1 = x - y - 2c$
$\alpha_2 = y - c$ it is a spanning set.
$\alpha_3 = c$

but $\underbrace{\underset{v_3}{\begin{bmatrix}3\\1\end{bmatrix}} = 2\cdot\underset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}} + 1\cdot\begin{bmatrix}1\\1\end{bmatrix}}$ v_2 ☆

3.3 Linear Independence

(1) if $v_1, v_2, \ldots, v_k$ span V and one of the v_i is a linear combination of the other $(k-1)$ then the other $(k-1)$ vectors span V.

ex) above $v_3 = 2v_1 + v_2$ ☆

but $2\mathbb{V}_1 + \mathbb{V}_2 - \mathbb{V}_3 = \mathbb{O}$

let $A = [\mathbb{V}_1\ \mathbb{V}_2\ \mathbb{V}_3]$ $\rightarrow$ $A \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \boxed{\mathbb{O}}$

(2) we have a linear combination $\boxed{\text{if and only if}}$ iff

there are scalars c_i (not all zero)

such that $c_1\mathbb{V}_1 + c_2\mathbb{V}_2 + \dots + c_k\mathbb{V}_k = \mathbb{O}$

as system of eqn's: let $A = [\mathbb{V}_1\ \mathbb{V}_2 \dots \mathbb{V}_k]$

$\mathbb{C} = [c_1\ c_2 \dots c_k]^T$

$\rightarrow$ $A\mathbb{C} = \mathbb{O}$ (homogeneous system)

Def $\boxed{\text{linear dependence}}$ if $A\mathbb{C} = \mathbb{O}$
where $A = [\mathbb{V}_1\ \mathbb{V}_2 \dots \mathbb{V}_k]$ has non-trivial soln

$\boxed{\text{linear independence}}$ if $A\mathbb{C} = \mathbb{O}$
has only trivial solution.

Note: $A\mathbb{C} = \mathbb{O}$

is $c_1\mathbb{V}_1 + c_2\mathbb{V}_2 + \dots + c_k\mathbb{V}_k = \mathbb{O}$

ex

$\mathbb{R}^n$ Vector Space given n-vectors.

$A = [V_1 \; V_2 \cdots V_n]$ is $n \times n$

$Ac = 0$ does it have non-trivial soln's?

check $\det(A) = 0 \to$ singular
$\to A^{-1}$ doesn't exist
$\to$ non-trivial solns

So $\boxed{V_i \text{ are linearly dep}}$

vs $\det(A) \neq 0 \to$ non-singular
$\to$ only trivial soln
$\to \boxed{V_i \text{ are linearly independent}}$

$\mathbb{R}^n$ Vector space have k-vectors $(k \neq n)$

$A = [V_1 \; V_2 \cdots V_k]$ is $n \times k$ (not square)

Solve: $Ac = 0$ and check if we have non-trivial solutions.

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ex are $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ linearly ind? dep?

Heart: $\alpha_1\begin{bmatrix}1\\0\\1\end{bmatrix}+\alpha_2\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ solns?

Solve by..

Aug. Matrix

$\left[\begin{array}{cc|c}1&1&0\\0&1&0\\1&1&0\end{array}\right]$

$\Downarrow$

$\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right]$ $\alpha_2=0$, $\alpha_1=0$

System of Eqn's

$\begin{cases}\alpha_1+\alpha_2=0\\ \alpha_2=0\\ \alpha_1+\alpha_2=0\end{cases} \Rightarrow \begin{array}{l}\alpha_1=0\\ \alpha_2=0\end{array}$

only trivial $(0,0)$ soln.

$\therefore$ $\boxed{\text{independent}}$

Other Vector Space?: $\left(\mathbb{R}^{m\times n},\ P_n,\ C[a,b]\right)$

Note: $c_1\mathbb{V}_1+c_2\mathbb{V}_2+\dots+c_k\mathbb{V}_k=\mathbb{O}$ is still the test

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$\mathbb{R}^{m\times n}$

ex $\mathbb{R}^{2\times 2}$; $\mathbb{V}_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\mathbb{V}_2=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\mathbb{V}_3=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

linearly ind/dep?

$c_1\mathbb{V}_1+c_2\mathbb{V}_2+c_3\mathbb{V}_3=\mathbb{O}$

Solve $c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$$\rightarrow \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = 0 \qquad c_2 = 0$

$c_3 = 0$ $\rightarrow$ only trivial sol.

$\therefore$ $\boxed{\text{independent}}$

$\mathbb{R}^{2\times 2}$; $V_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, V_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

ind/dep?

Solve $c_1 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} c_1 + c_2 + c_3 & c_1 + c_4 \\ 2c_1 + c_3 + c_4 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_4 = 0 \\ 2c_1 + c_3 + c_4 = 0 \\ c_1 + c_2 = 0 \end{cases}$$

$\rightarrow$ solve? (finish!)

P_n Vector space

ex P_4; $v_1 = 2+x$, $v_2 = x^2+x^3$, $v_3 = 1+x^2$, $v_4 = x-x^3$

ind/dep?

Solve: $c_1(2+x)+c_2(x^2+x^3)+c_3(1+x^2)+c_4(x-x^3) = 0+0x+0x^2+0x^3$

$(2c_1+c_3)+(c_1+c_4)x+(c_2+c_3)x^2+(c_2-c_4)x^3 = 0+0x+0x^2+0x^3$

So $\begin{cases} 2c_1+c_3=0 \\ c_1+c_4=0 \\ c_2+c_3=0 \\ c_2-c_4=0 \end{cases}$ $\xrightarrow{\text{Solve!}}$ (finish!)

$C[a,b]$ New notation $C^{[n]}[a,b]$ all cont. functions over $[a,b]$

and 1st deriv., 2nd deriv., ⋮ nth deriv } are also cont.

ex $C[-\pi,\pi]$; $v_1 = \cos x$, $v_2 = 1$, $v_3 = \sin^2(x/2)$

ind/dep?

Solve $c_1\cos x + c_2\cdot 1 + c_3\sin^2(x/2) = 0$

$-c_1\sin x + 0 + 2c_3\sin(x/2)\cos(x/2)(1/2) = 0$

$-c_1\cos x + 0 + [\text{deriv of } \uparrow] = 0$