# Math 511

Q5/ 3.2 5a) $P_4$ is $S = \{$all poly's in $P_4$ that are even deg$\}$ a subspace?

Background

1) anything in $P_4$ is $p(x) = a + bx + cx^2 + dx^3$
coeff are Reals

2) $\mathcal{O}$ of $P_4$ $z(x) = 0 + 0x + 0x^2 + 0x^3$

3) even degree polynomial
→ ~~highest power is even?~~
→ all powers that have a non-zero coeff are even? $1 \cdot x^0 + x^2$
→ sym about y-axis

4) Set of all poly that have a degree thats even

degree to a poly = highest power

$\boxed{\text{highest power is even}}$

a) $z(x) = 0 + 0x + 0x^2 + 0x^3$ degree is 0

b) $\alpha p(x) = \alpha a^0 + \alpha bx^1 + \alpha cx^2 + \alpha dx^3$

$\deg(p(x)) = \deg(\alpha p(x))$

No c) $p(x) + g(x)$ counter example

ex $(1 + x + x^2) + (1 + x - x^2) = 2 + 2x$

$\boxed{\text{Linear Ind}}$ for $V$ with $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \ldots, \mathbb{V}_k$

if $c_1\mathbb{V}_1 + c_2\mathbb{V}_2 + \cdots + c_n\mathbb{V}_n = \vec{0}$

a) has only trivial soln $(c_1=0, c_2=0, \ldots, c_k=0) \rightarrow \boxed{\text{ind}}$

b) has a non-trivial soln $\rightarrow \boxed{\text{dep}}$

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$\boxed{\text{ex}}$ $C[a,b]$ with $f_1, f_2, \ldots, f_n$

check ind/dep

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

ex $f_1 = 1$ $f_2 = x$ $f_3 = \sin x$ $f_4 = \sinh(x)$

$C[0,1]$

$$c_1 + c_2 x + c_3 \sin x + c_4 \sinh(x) = 0$$

can't solve this!

restrict the problem

$C^{(n-1)}[a,b]$ with $f_1, f_2, \ldots, f_n$

check ind/dep

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

1st deriv. $\rightarrow c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$

$\vdots$

$(n-1)^{\text{st}}$ deriv $c_1 f_1^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0$

Matrix of function expressions $\rightarrow$

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ \vdots & & \ddots & \vdots \\ f_1^{(n-1)}(x) & & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

**ind.?:** $x \in [a,b]$ if any $x_0$ exists

such that
$$\begin{bmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ \vdots & & & \\ f_1^{(n-1)}(x_0) & \cdots & \cdots & f_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has only trivial soln. $n \times n$

b/c it's $n \times n$ we can use $\det(\ )$ $\to \neq 0$ ind, $\to = 0$ dep

**Def** $\det \left( \begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \right) =$ function of $x$.

**Wronskian** $W[f_1, f_2, \ldots, f_n](x) = W(x)$

if $W(x)$ is ever $\neq 0$ for any $x \in [a,b]$

then $f_i$ are linear ind.!

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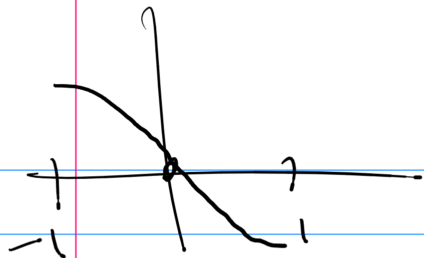
**Ex** $C^{(2)}[-1,1]$ $f_1 = 1$, $f_2 = x$, $f_3 = \sin(x)$

$$W(x) = \begin{vmatrix} 1 & x & \sin x \\ 0 & 1 & \cos x \\ 0 & 0 & -\sin x \end{vmatrix} = -\sin x$$



$W(1) = -\sin(1) \neq 0$

$\therefore$ $f_i$ are ind.

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$f_1 = \sin x \quad f_2 = \cos x$

$$W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$W(x) = -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x) = -1 \neq 0$

so $\boxed{\text{linearly ind.}}$

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**3.4** Minimal spanning set has

(1) $\mathrm{Span}(v_1, v_2, \ldots, v_n) = V$

(2) $v_i$ are linearly ind.

**Def** $v_1, v_2, \ldots, v_n$ are a $\boxed{\text{basis of } V}$ iff

(1) $v_i$ are linearly ind.

(2) $\mathrm{Span}(v_1, v_2, \ldots, v_n) = V$

**Thm** if $\mathrm{Span}(v_1, v_2, \ldots, v_n) = V$

any $m$-vectors, $m > n$, are linearly dep.

**Corollary** if $\left.\begin{array}{l} \mathrm{Span}(v_1, v_2, \ldots, v_n) = V \\ \mathrm{Span}(u_1, u_2, \ldots, u_m) = V \end{array}\right\}$ both $v_i$, $u_i$ are basis

$\Rightarrow n = m$

b/c number of vectors in a basis of V is uniq. (the vectors though are not)

let's use that uniq. number → $\boxed{\text{call it V's dimension}}$

ex $\mathbb{R}^3$ basis #1 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

basis #2 $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\dim(\mathbb{R}^3) = 3$$

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terms $\dim(V) = n$

① $V = \{0\}$ define $\dim(\{0\}) = 0$

② $\dim(V) = n$ and $n$ is a finite number

ex $\dim(\mathbb{R}^3) = 3$

$\dim(\mathbb{R}^{2\times 3}) = 6$

call $V$ finite dimensional.

③ otherwise $V$ is infinite dimensional.

Standard Basis

$\mathbb{R}^n$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$P_n \quad p_1 = 1, \; p_2 = x, \; p_3 = x^2, \; \cdots, \; p_n = x^{n-1}$

$\mathbb{R}^{2\times 2} \quad M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad M_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$