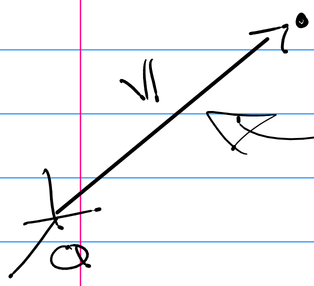
Math 511

V is a vector space given a basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$



$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

$$\mathbf{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_B \to \text{basis of } \mathbf{b}_i$$

Note: Standard basis of $\mathbb{R}^n$: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$

Let $E = [\mathbf{e}_1\, \mathbf{e}_2 \cdots \mathbf{e}_n]$ ← Note this is simply I

E is standard basis

$B = [\mathbf{b}_1\, \mathbf{b}_2 \cdots \mathbf{b}_n]$

Conversion from basis B to Standard basis

$$[\mathbf{x}]_E = B[\mathbf{x}]_B$$

Conversion from Standard to basis B

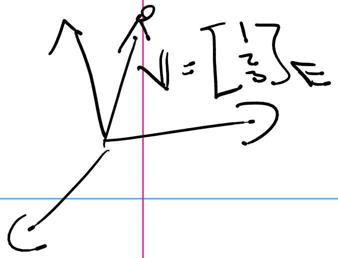
$$[\mathbf{x}]_B = B^{-1}[\mathbf{x}]_E$$

Ex $\mathbb{R}^3$ $\quad E = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} \quad B = \begin{bmatrix} 1&1&1\\0&0&1\\0&1&1 \end{bmatrix}$

Find B^{-1} $\quad [B\,|\,I] = \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\0&0&1&0&1&0\\0&1&1&0&0&1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\0&1&1&0&0&1\\0&0&1&0&1&0 \end{array}\right]$

$$\to \left[\begin{array}{ccc|ccc} 1&1&0&1&-1&0\\0&1&0&0&-1&1\\0&0&1&0&1&0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1&0&0&1&0&-1\\0&1&0&0&-1&1\\0&0&1&0&1&0 \end{array}\right]$$

$$\therefore B^{-1} = \begin{bmatrix} 1&0&-1\\0&-1&1\\0&1&0 \end{bmatrix} \qquad B = \begin{bmatrix} 1&1&1\\0&0&1\\0&1&1 \end{bmatrix}$$



$$[\vec{v}]_B = B^{-1}[\vec{v}]_E$$

$$[\vec{v}]_B = \begin{bmatrix}1&0&-1\\0&-1&1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}_E = \begin{bmatrix}-2\\1\\2\end{bmatrix}_B$$

$$\boxed{v = \begin{bmatrix}1\\2\\3\end{bmatrix}_E} \qquad \boxed{v = \begin{bmatrix}-2\\1\\2\end{bmatrix}_B} = (-2)\begin{bmatrix}1\\0\\0\end{bmatrix} + (1)\begin{bmatrix}1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}_B = \begin{bmatrix}?\\ \\0\end{bmatrix}_E \quad \text{so} \quad [\vec{v}]_E = B\begin{bmatrix}1\\2\\3\end{bmatrix}_B$$

$$[\vec{v}]_E = \begin{bmatrix}1&1&1\\0&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}_B = \begin{bmatrix}6\\3\\5\end{bmatrix}_E$$

Given two non-stand bases

$$U = \{u_1, u_2 \dots u_n\} \qquad V = [v_1, v_2 \dots v_n]$$

Q5.1

① How to convert from basis U to basis V?

② How to convert from basis V to basis U?

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① Convert from U to V: $[\vec{v}]_U \to [?]_V$

Idea go from U to E and E to V

$$[\vec{v}]_E = U[\vec{v}]_U \quad \text{and} \quad [\vec{v}]_V = V^{-1}[\vec{v}]_E$$

So $\boxed{[\,c\,]_V = V^{-1} U [\,c\,]_U}$ $S = V^{-1}U$ transition matrix from U to V

part 2 V to U?

$\boxed{[\,c\,]_U = U^{-1} V [\,c\,]_V}$ S^{-1} transition matrix from V to U

P_3 standard is $\{1, x, x^2\} = \{1 + 0x + 0x^2,\ 0 + 1x + 0x^2,\ 0 + 0x + 1x^2\}$

B is $\{1+x,\ x^2-2,\ 3\} = \{1 + 1x + 0x^2,\ -2 + 0x + 1x^2,\ 3 + 0x + 0x^2\}$

Standard $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

given $p(x) = 4 - x + 2x^2 \Rightarrow P = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}_E$

$P = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}_B = ?(1+x) + ?(x^2-2) + ?(3)$

Know $[\,c\,]_B = B^{-1}[\,c\,]_E = B^{-1}\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}_E$

Use $\left[B \,\middle|\, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right] \xrightarrow[\text{ops}]{\text{row}} \left[I \,\middle|\, B^{-1}\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right]$

So $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & -2 & 3 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & 3 & 5 \end{array}\right]$

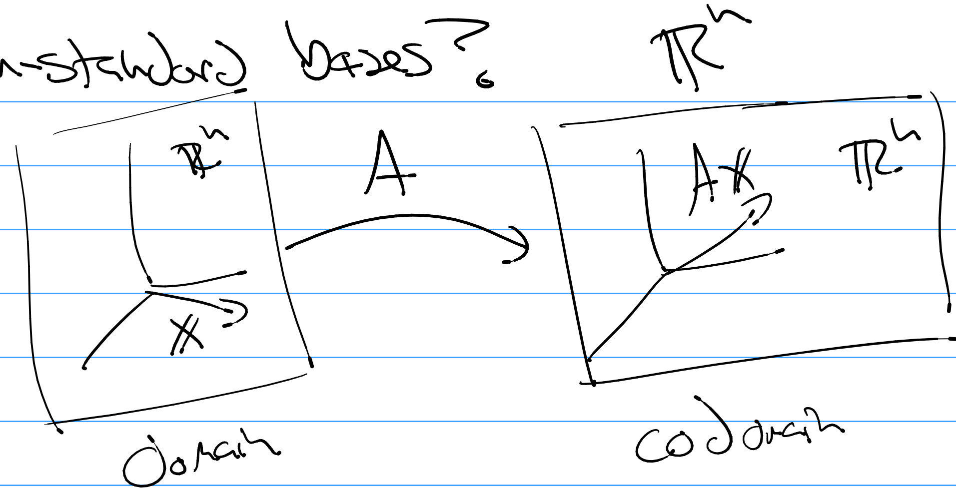
$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ so $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_B = P$

So $4 - x + 2x^2 = (-1)(1+x) + 2(x^2-2) + 3(3)$

(ex) Why use non-standard bases?

A is $n \times n$

(1)

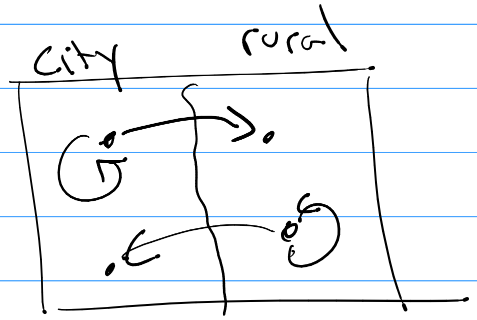


(2) $x_0, \; x_1 = A x_0, \; x_2 = A x_1, \; x_3 = A x_2, \; \ldots$

Markov process

seq of vectors $x_0, x_1, x_2, \ldots \to$ $\boxed{\text{Markov chain}}$

(ex)



$$x = \begin{bmatrix} \text{city} \\ \text{rural} \end{bmatrix}$$

$$A x = \begin{array}{cc} & \begin{array}{cc} c & \quad r \end{array} \\ \begin{array}{c} c \\ r \end{array} & \begin{bmatrix} c \to c & r \to c \\ c \to r & r \to r \end{bmatrix} \end{array} \begin{bmatrix} \text{city} \\ \text{rural} \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 30\% \\ 70\% \end{bmatrix} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$$

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix} \qquad x_1 = A x_0$$

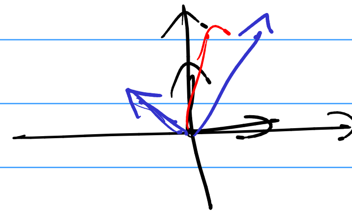
$$x_1, x_2, \ldots, x_n \to \begin{bmatrix} .25 \\ .75 \end{bmatrix}$$

Study $A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}$

See $A x^* = \lambda x^*$

(to be done later: eigen value/vectors)

$$A \overset{u_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} = 1 \cdot \overset{u_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$



$$A \overset{u_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = .92 \cdot \overset{u_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

but $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ b/c lin. ind., they are a basis!

$$U = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$$

$$x = [c]_U = c_1 u_1 + c_2 u_2$$

$$x_0 = \begin{bmatrix} .3 \\ .7 \end{bmatrix}_E = \begin{bmatrix} .25 \\ -.05 \end{bmatrix}_U = .25\, u_1 - 0.05\, u_2$$

$$A x_0 = A(.25\, u_1 - 0.05\, u_2)$$

$$A x_0 = .25\, A u_1 - 0.05\, A u_2$$

$$x_1 = A x_0 = .25\, u_1 - 0.05(.92)\, u_2$$

$$x_2 = A x_1 = .25\, u_1 - 0.05\,(.92)^2\, u_2$$

$$\vdots$$

$$x_n = A x_{n-1} = .25\, u_1 - 0.05\, \underbrace{(.92)^n}_{\to 0}\, u_2$$

as $n \to \infty$ $\quad x_n \to .25\, u_1 = .25 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} .25 \\ .75 \end{bmatrix}$