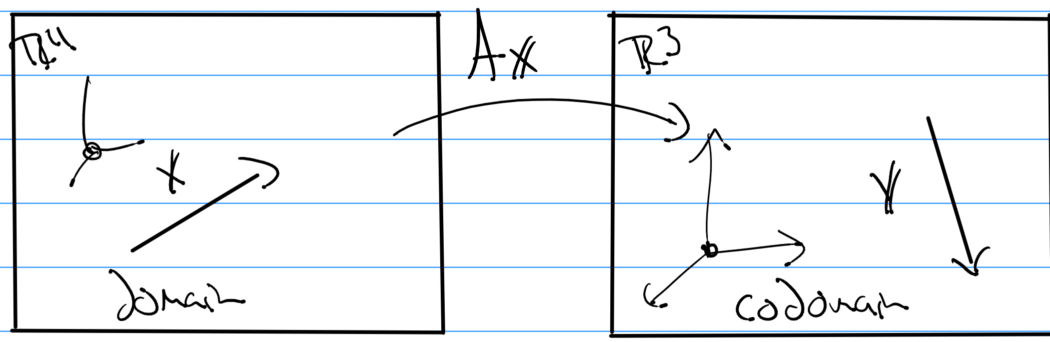
Math 511

Ex $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 3 & 1 & 2 \\ 1 & 5 & 5 & 0 \end{bmatrix}$ size: 3×4

$\rightarrow \underset{3\times4}{A}\,\underset{4\times1}{\mathbf{x}} = \underset{3\times1}{\mathbf{y}}$



rowspace of A = Span(A's rows)

= Span([1 1 2 -1], [-1 3 1 2], [1 5 5 0])

$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 3 & 1 & 2 \\ 1 & 5 & 5 & 0 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 4 & 3 & 1 \\ 0 & 4 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5/4 & -5/4 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

b/c row ops = linear combos of A's rows

$\rightarrow$ Span(A's rows) = Span(U's rows)

$\therefore$ row space of A = Span([1 0 5/4 -5/4], [0 1 3/4 1/4])

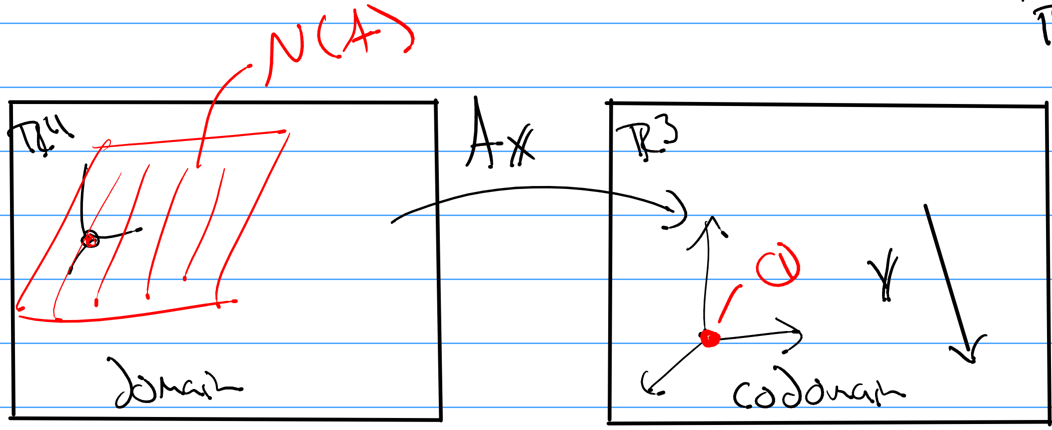
$\rightarrow$ dim(row space of A) = 2 = # of lead vars.

call rank(A) = 2

$A_{m\times n}$ $\quad N(A) = \{$ all $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \underset{\mathbb{R}^m}{\mathbf{0}}\}$

$\underset{n\times 1}{A\mathbf{x}} = \underset{m\times 1}{\mathbf{b}}$

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 3 & 1 & 2 \\ 1 & 5 & 5 & 0 \end{bmatrix}$$



to find ← get a basis for N(A) subspace

Solve $A\mathbf{x} = \mathbf{0}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ -1 & 3 & 1 & 2 & 0 \\ 1 & 5 & 5 & 0 & 0 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \begin{array}{ccccc} x_1 & x_2 & \text{free's} & & \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & 5/4 & -5/4 & 0 \\ 0 & \boxed{1} & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

let $x_3 = \alpha \quad x_4 = \beta$

$x_2 = -3/4\,\alpha - \frac{1}{4}\beta$

$x_1 = -5/4\,\alpha + 5/4\,\beta$

$$\therefore \mathbf{x} = \begin{bmatrix} -5/4\,\alpha + 5/4\,\beta \\ -3/4\,\alpha - 1/4\,\beta \\ 1\alpha + 0\beta \\ 0\alpha + 1\beta \end{bmatrix}$$

$$\text{So } \mathbf{x} = \alpha \begin{bmatrix} -5/4 \\ -3/4 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5/4 \\ -1/4 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore N(A) = \text{Span}\left(\begin{bmatrix} -5/4 \\ -3/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/4 \\ -1/4 \\ 0 \\ 1 \end{bmatrix} \right)$$

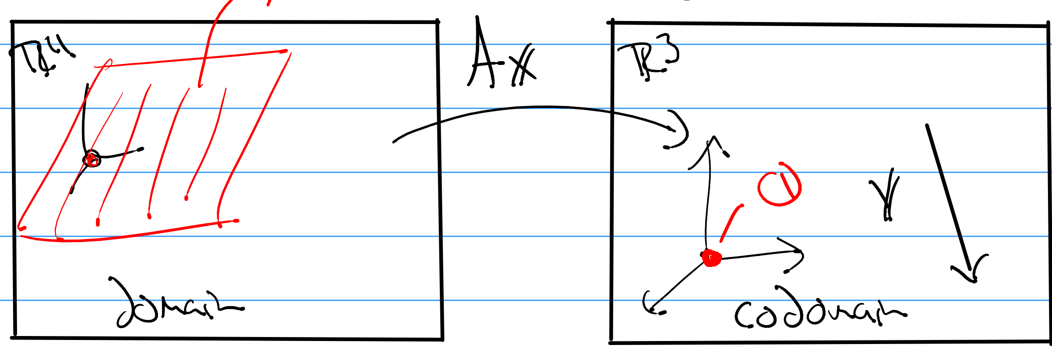
dim $(N(A)) = 2$ = # of free variables

Def nullity of A = $\dim(N(A))$ = # of free vars.

Thm b/c vars = # of lead + # of free

$\Rightarrow$ $\text{rank}(A)$ + nullity of A = # of vars

ex $N(A) = \text{Span}\left(\begin{bmatrix} -5/4 \\ -3/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/4 \\ -1/4 \\ 0 \\ 1 \end{bmatrix}\right)$



Column Space of A = Span(A's columns)

ex $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 3 & 1 & 2 \\ 1 & 5 & 5 & 0 \end{bmatrix}$ Col Space of A = $\text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right)$

can we pair down the dependent vectors?

based on $A\mathbf{x} = \mathbf{b}$ (solve?)

1. $\mathbf{b}$ must be in col space of A.
2. uniq. soln of $A\mathbf{x} = \mathbf{b}$ means A's cols must be a basis of $\mathbb{R}^n$

Dependency for A's Columns?

① Consider dependency of U's Columns.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 3 & 1 & 2 \\ 1 & 5 & 5 & 0 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} U = \begin{bmatrix} 1 & 0 & 5/4 & -5/4 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u|_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u|_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad u|_3 = \begin{bmatrix} 5/4 \\ 3/4 \\ 0 \end{bmatrix} = \tfrac{5}{4} u|_1 + \tfrac{3}{4} u|_2$$

So $\boxed{u|_3 = \tfrac{5}{4} u|_1 + \tfrac{3}{4} u|_2}$

$\boxed{u|_4 = -\tfrac{5}{4} u|_1 + \tfrac{1}{4} u|_2}$

Consider:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix} \xrightarrow[\text{op}]{\text{row}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ & & \end{bmatrix}$$

$a|_3 = a|_1 + a|_2$

$$2a + 2b = 2c \qquad c = a + b$$

$$+(d + e = f) \qquad f = e + d$$

So row ops change columns $\boxed{\text{but}}$ they <u>do not</u> change <u>dependency</u>.

$\therefore$ $u|_3 = \tfrac{5}{4} u|_1 + \tfrac{3}{4} u|_2$
$u|_4 = -\tfrac{5}{4} u|_1 + \tfrac{1}{4} u|_2$
u's dep.

$\longrightarrow$ $\boxed{\begin{aligned} a|_3 &= \tfrac{5}{4} a|_1 + \tfrac{3}{4} a|_2 \\ a|_4 &= -\tfrac{5}{4} a|_1 + \tfrac{1}{4} a|_2 \end{aligned}}$

col space of A = Span(A's columns)

= b/c a_3, a_4 are dependent

So col space of A = Span(a_1, a_2)

$= \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \right)$

dim (col space of A) = 2 = # of lead vars.

basis of col. space of A = $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \right\}$

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rank(A) = # of leads = dim(row space of A)

= dim(col. space of A)

nullity of A = # of free = dim($N(A)$)

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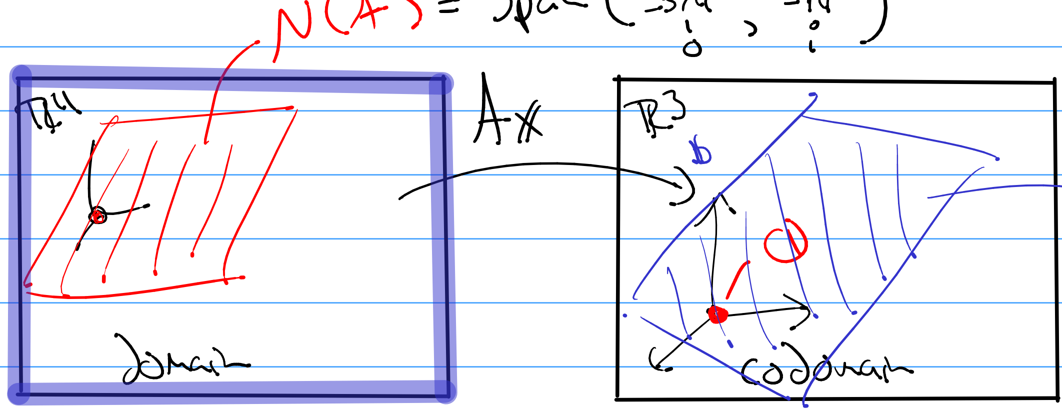
ex

$N(A) = \text{Span}\left(\begin{pmatrix} -5/4 \\ -3/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5/4 \\ -1/4 \\ 0 \\ 1 \end{pmatrix} \right)$



range
||
col space of A
||
$\text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \right)$

College Alg.

$y = x^2$