Math 511

Q's/

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \to u = [\ \ ]$$

\# lead vars = rank(A)

\# free vars = nullity(A)

total vars = n = rank(A) + nullity(A)

3.6 #10 a) given rank(A) = n all lead vars so 0 free vars.

b/c no free vars → $m \geq n$ system is det. or over det.

also given $\boxed{A\mathbf{c} = A\mathbf{d}}$ b/c no free vars. 1 soln or 0 solns

says we have a soln.

b/c we have only 1 or 0 → we have 1.

∴ $\mathbf{c} = \mathbf{d}$

b) rank(A) < n → free vars (says we have ∞ solns or 0 solns)

b/c $A\mathbf{c} = A\mathbf{d}$

says we have a soln → ∞ solns.

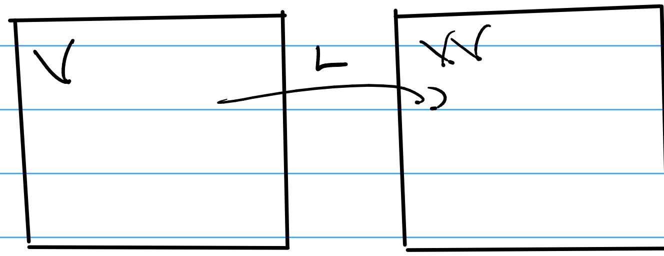
(we can show these ∞ solns by

$$A\mathbf{c} = A\mathbf{d} \to A(\mathbf{c} - \mathbf{d}) = 0$$

so $\alpha(\mathbf{c} - \mathbf{d})$ is a soln.

Ch 4 Linear Transformations / Operators

$L : V \to W$



$L(x) = y$

$x \in V$, $y \in W$

Property of L to check ...

Show ① $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$

or

Show ① $L(v_1 + v_2) = L(v_1) + L(v_2)$

② $L(\alpha v_1) = \alpha L(v_1)$

Features of Interest

① $\ker(L) = \{ v \in V \mid L(v) = 0_W \}$

② range of L = $L(V) = \{ w \in W \mid$ there was a $v \in V$, $L(v) = w \}$

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ex $L : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \boxed{L(x) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}} \to L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

is L a linear transform? $\quad v_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

① Show $L(v_1 + v_2) = L(v_1) + L(v_2)$

$$L(v_1 + v_2) = L\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = L(v_1) + L(v_2)$$

(i) Show $L(\alpha \mathbb{V}_1) = \alpha L(\mathbb{V}_1)$

$$L(\alpha \mathbb{V}_1) = L\begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{pmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} = \alpha L(\mathbb{V}_1)$$

true!

$$\text{Ker}(L) = \{ \mathbb{V} \in \mathbb{V} \mid L(\mathbb{V}) = 0_W \}$$

$\mathbb{V} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ all $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

so $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad a = 0$, $b = 0$, $c =$ anything

Soln. $\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}$

$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore \text{Ker}(L) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

Range of L $\quad L(\mathbb{V}) = \{ \mathbb{W} \in W \mid$ there is a $\mathbb{V} \in V$ $L(\mathbb{V}) = \mathbb{W} \}$

if $L(\mathbb{V}) = L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \mathbb{W}$

a, b, c are any #'s

$$\mathbb{W} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$L(\mathbb{V}) = \text{Span}\left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right)$$

4.2 from 4.1 we know if $A_{m \times n}$ then $A\mathbb{x} = \mathbb{y}$ is a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^m$

Given $L : \mathbb{R}^n \to \mathbb{R}^m$ can we find an $A_{m \times n}$ such that $L(\mathbb{x}) = A\mathbb{x}$

ex above we had $L(\mathbb{x}) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ was a linear trans.

can we find A so that

$$L(\mathbb{x}) = A\mathbb{x}$$

<u>Note:</u>

$$\mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_E = \overbrace{x_1 e_1 + x_2 e_2 + \cdots + x_n e_n}^{\text{is a linear combo}}$$

$$L(\mathbb{x}) = L(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n)$$

$$= \begin{bmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

So $L(\mathbb{x}) = \begin{bmatrix} \boxed{L(e_1)} & L(e_2) & \cdots & L(e_n) \end{bmatrix} \mathbb{x}$

$$= A\mathbb{x}$$

where $A = \begin{bmatrix} L(e_1) & L(e_2) & \cdots & L(e_n) \end{bmatrix}$

(ex) $L(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ find A so that $L(\mathbf{x}) = A\mathbf{x}$

$$A = \left[L\begin{pmatrix}1\\0\\0\end{pmatrix} \; L\begin{pmatrix}0\\1\\0\end{pmatrix} \; L\begin{pmatrix}0\\0\\1\end{pmatrix} \right]$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

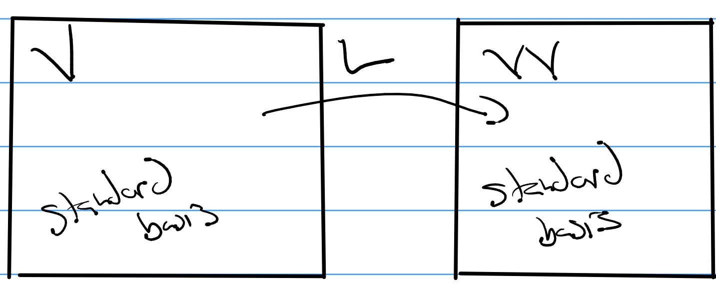
(ex) $L\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}$

↑ of orthog!

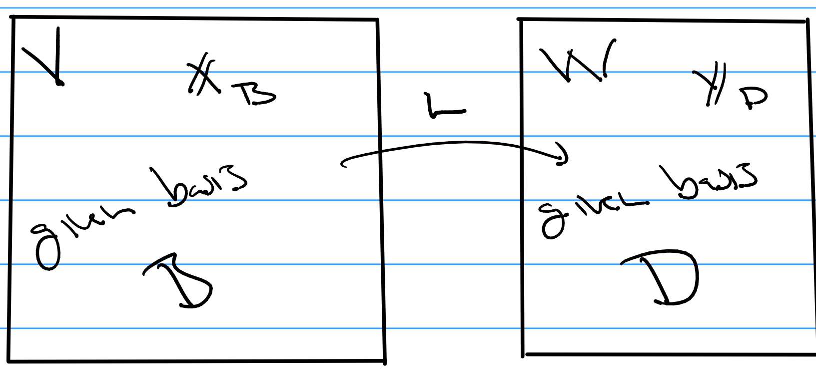
Note: given $L(\mathbf{x})$ as a standard basis transform

$$\mathbf{y}_E = L(\mathbf{x}_E) \to [\mathbf{y}]_E = A[\mathbf{x}]_E$$

and $A = [L(\mathbf{e}_1) \; L(\mathbf{e}_2) \ldots L(\mathbf{e}_n)]$



(b)



$$\mathbf{y}_D = \begin{pmatrix} ? \\ 0 \end{pmatrix} \mathbf{x}_B$$

Use $$[Y]_D = D^{-1} A \big(\overbrace{B\,[X]_B}^{\text{in basis standard}}\big)$$

transform L in standard basis

$$[Y]_D = \big(\underbrace{D^{-1} A B}_{}\big)\,[X]_B$$

Matrix of $L : V \to W$ in basis B to basis D

Note: $AB = [\,L(b_1)\ \ L(b_2)\ \cdots\ L(b_n)\,]$

$L : \mathbb{R}^2$ to $\mathbb{R}^3$

$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

$\mathbb{R}^2$ basis is $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$\mathbb{R}^3$ basis is $D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$[Y]_D = (\ ?\ )\,[X]_B$$

$$= D^{-1} A B\, [X_B]$$

$$A = \left[\,L\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ \ L\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[Y]_D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \underbrace{\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\right)}_{= \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix}} [X]_B$$