Math 511

~~Q's~~ / 4.2 $C[a,b]$

$$S = \text{Span}(e^x, xe^x, x^2e^x)$$

ex P_3 $p(x) = 2x^2 + x$ as a vector $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\begin{matrix} \leftarrow x^2 \\ \leftarrow x \\ \leftarrow \text{const.} \end{matrix}$

Vectors for $\text{Span}(e^x, xe^x, x^2e^x) = S$

$$f(x) = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}\begin{matrix} \leftarrow e^x \\ \leftarrow xe^x \\ \leftarrow x^2e^x \end{matrix}$$

ex $f(x) = 2e^x - xe^x + 3x^2e^x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

so any $f(x) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boxed{a e^x + b\underline{xe^x} + cx^2e^x}$

$L: V \to W$ for the problem is $D : \underset{S}{\cancel{C[a,b]}} \to \underset{S}{\cancel{C[a,b]}}$

D: 1st deriv.

$$D\begin{bmatrix} a \\ b \\ c \end{bmatrix} = f'(x) = ae^x + b(e^x + xe^x) + c(2xe^x + x^2e^x)$$
$$= (a+b)e^x + (b+2c)xe^x + cx^2e^x$$
$$= \begin{bmatrix} a+b \\ b+2c \\ c \end{bmatrix}$$

SO basis: $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = [e^x, xe^x, x^2e^x]$

$$D\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+2c \\ c \end{bmatrix} \leftarrow$$

$$A = \left[\underline{D\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \; D\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; D\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 1 \\ 1 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 2 \\ 1 \end{matrix}\right]$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{d}{dx}[xe^x] = e^x + xe^x = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$D\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{d}{dx}[e^x] = e^x = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$D\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{d}{dx}[x^2e^x] = 2xe^x + x^2e^x = \begin{bmatrix}0\\2\\1\end{bmatrix}$$

So

as example

$$g(x) = 7e^x + 2xe^x - \pi x^2 e^x$$

$$\frac{d}{dx} g(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}7\\2\\-\pi\end{bmatrix} = g'(x)$$

Matrix Representation of Linear Transform $L: V \to W$

depends on the basis of your vector spaces

ex) use basis B for V, basis D for W

a) $y = L(x)$ (standard basis for both V, W)

$$A = [L(e_1)\ L(e_2)\ ..\ L(e_n)]$$

b) $[y]_D = L([x]_B)$

$$[y]_D = \boxed{D^{-1} A B}[x]_B$$

Matrix representation of L using bases B for V, D for W

Special Case: $L: V \to V$ given standard, given basis $\mathcal{B}$

a) $A = [\,L(e_1)\ \ L(e_2)\ \ldots\ L(e_n)\,]$ standard basis representation of L

b) $[y]_B = B^{-1} A B [x]_B$

So for $L: V \to V$

1. A is the matrix representation of L using standard basis
2. $B^{-1} A B$ is the matrix representation of L using basis $\mathcal{B}$.

let $M = B^{-1} A B$

$\boxed{\text{Def}}$ given M_1, M_2 are both $n \times n$

M_1, M_2 are $\boxed{\text{similar}}$ if there is some non-singular (it has an inverse) matrix S such that $M_2 = S^{-1} M_1 S$

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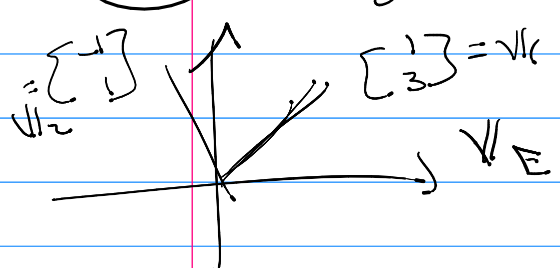
ex Eigen value / Vector idea.



transform

$$[y]_E = V^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V [v]_E$$

$\boxed{\text{thm}}$ Q is a vector space and given two bases B_1, B_2

$$[X]_E = B_1 [X]_{B_1}, \quad [X]_E = B_2 [X]_{B_2}$$

convert $[X_{B_1}]$ to $[X_{B_2}]$

$$[X]_{B_2} = \underbrace{B_2^{-1} B_1}_{= S} [X]_{B_1}$$

S converts basis B_1 to B_2

$\boxed{\text{Now}}$ given $\boxed{A}$ the matrix rep. of L in basis B_2

$$[Y]_{B_1} = S^{-1} A S [X]_{B_1}$$

the matrix is basis B_1 is $B = S^{-1} A S$

$$[Y]_{B_1} = \underbrace{\overset{S^{-1}}{B_1^{-1} B_2} \underbrace{\overset{\text{lin. trans. in basis}}{A_{B_2}} \underbrace{\overset{S}{B_2^{-1}} \underbrace{B_1 [X]_{B_1}}_{\text{standard}}}_{\text{base } B_2}}_{\text{transformed but is still in basis } B_2}}_{\text{standard basis}}$$

(ex) Subspace of $C[a,b]$

$$S = \text{span}(\underbrace{1, e^x, e^{-x}}_{B_1}) \qquad 1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ e^x = \begin{bmatrix}0\\1\\0\end{bmatrix},\ e^{-x} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$S = \text{span}(1, \cosh x, \sinh x) = \text{span}\left(1, \frac{e^x + e^{-x}}{2}, \frac{e^x - e^{-x}}{2}\right)$$

$$B_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 1/2\\ 0 & 1/2 & -1/2\end{bmatrix} \text{ vs } \boxed{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1\end{bmatrix}} \neq B_2!$$

make same span so

$$B_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad B_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1\end{bmatrix}$$

on $\text{span}(1, e^x, e^{-x}) = S$

$\boxed{\text{Consider}}$ $\quad L: S \to S$, L is the 1^{st} deriv.

$$L\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \frac{d}{dx}\left[a + be^x + ce^{-x}\right]$$

$$= 0 + be^x - ce^{-x} = \begin{bmatrix}0\\b\\-c\end{bmatrix}$$

$$A_E = \left[\underset{\begin{smallmatrix}1\\0\\0\end{smallmatrix}}{L(e_1)}\ \ \underset{\begin{smallmatrix}0\\1\\0\end{smallmatrix}}{L(e_2)}\ \ \underset{\begin{smallmatrix}0\\0\\1\end{smallmatrix}}{L(e_3)}\right] = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

L with respect to B_2? $\quad B = \boxed{B_2^{-1} A_E B_2}$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$B = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{ccc} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} \leftarrow 1 \\ \leftarrow \cosh \\ \leftarrow \sinh \end{array} \\ \begin{array}{ccc} \frac{d}{dx}[1] & \frac{d}{dx}(\cosh x) & \frac{d}{dx}(\sinh x) \end{array} & \end{array}$$