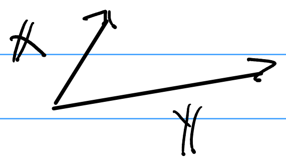
# Math 511

$V$ is a vector space (objects, $v_1 + v_2$, $\alpha v$ satisfy Axioms)

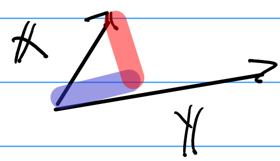
→ Subspaces, dim, $Ker(L)$, $L(V)$ a $L: V \to W$

Study structures of the space itself?

① length (magnitude $\mathbb{R}^2, \mathbb{R}^3$

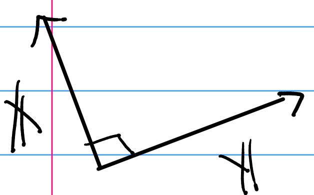



② "same direction" (contribution)

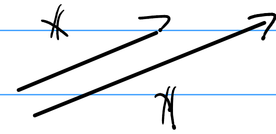



no contribution for $x$ in $y$'s direction

- orthogonal
- $\boxed{\text{perpendicular}}$ $\mathbb{R}^2, \mathbb{R}^3$




some contribution of $x$ in $y$'s direction

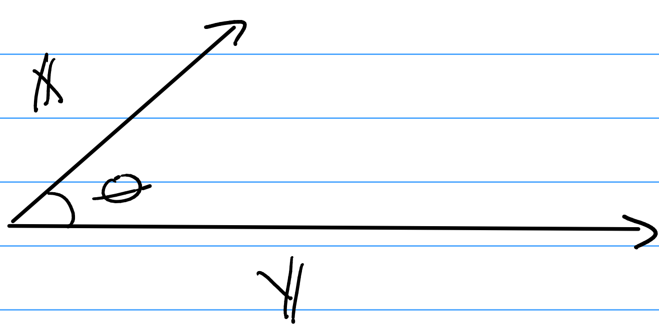



in-line contribution same direction

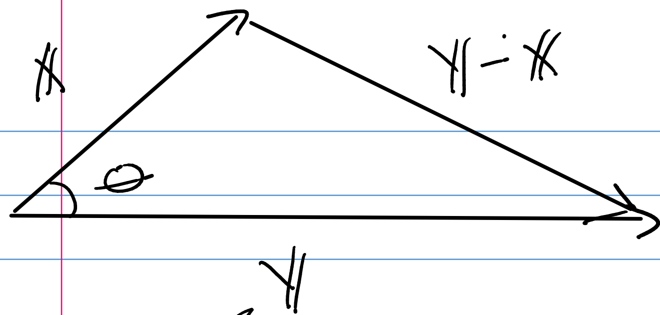
- $\boxed{\text{parallel}}$ $\mathbb{R}^2, \mathbb{R}^3$

## $\boxed{5.1}$ Orthogonality (length in $\mathbb{R}^n$

$\mathbb{R}^2 / \mathbb{R}^3$ with Euclidean Geometry




want a "contribution"/"direction" measure → let's use $\theta$




Law of cosines

$$\|y-x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

Need $\|v\| = ?$

Note: $x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ — scalar product

$\mathbb{R}^2 | \mathbb{R}^3$ $\quad x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3$

so $\quad x^T x = x_1^2 + x_2^2 + x_3^2$

length of $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$ $\quad \mathbb{R}^3$

$$\boxed{\|x\| = \sqrt{x^T x} \rightarrow \|x\|^2 = x^T x}$$

Now that we have a way to find lengths we can go back to...

$$\|y-x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\boxed{\cos\theta}$$

this can now be solved for

$$\rightarrow \cos\theta = \frac{1}{2\|x\|\|y\|}\left[\|x\|^2 + \|y\|^2 - \|y-x\|^2\right]$$

$$\rightarrow \cos\theta = \frac{1}{2\|x\|\|y\|}\left[x^T x + y^T y - (y-x)^T(y-x)\right]$$

$$\rightarrow \cos\theta = \frac{1}{2\|x\|\|y\|}\left[\cancel{x^T x} + \cancel{y^T y} - \left(\cancel{y^T y} - \underline{y^T x} - \underline{x^T y} + \cancel{x^T x}\right)\right]$$

$\to \cos\theta = \dfrac{x^T y}{\|x\| \, \|y\|} \to$ unit vectors $u, v$

$\to \boxed{\cos\theta = u^T v}$

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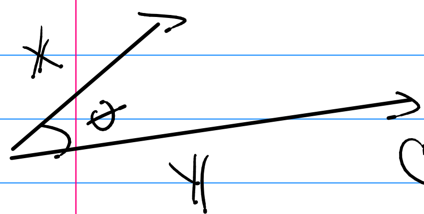
So in $\mathbb{R}^2, \mathbb{R}^3$ the scalar product $x^T y$ can be used for metrics of length and angles

directions or contributions



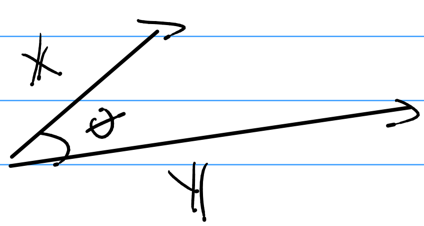

(1) $\|x\|^2 = x^T x$

(2) $\cos\theta = \dfrac{x^T y}{\|x\| \, \|y\|}$

(3) b/c $-1 \le \cos\theta \le 1 \to |\cos\theta| \le 1$

$\to |x^T y| \le \boxed{\|x\| \, \|y\|}$

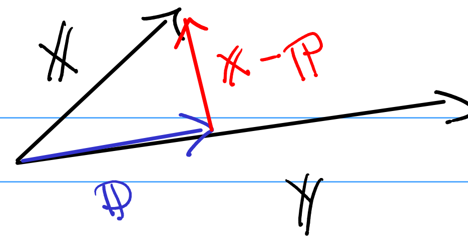
Cauchy-Schwartz inequality




(4) b/c @ $\pi/2$ (90°) $\cos(\pi/2) = 0$

Def $\to$ $x, y$ are orthogonal ($x \perp y$) if $x \neq 0$, $y \neq 0$, $x^T y = 0$

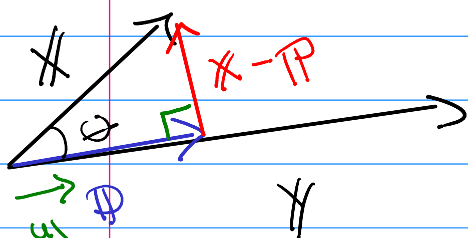
$\boxed{Q}$ what if $x^T y \neq 0$




x has two parts. (i) p is the part of x in y's direction

(ii) x − p is the part of x ⊥ to y.

$\boxed{\text{Need to find } p}$

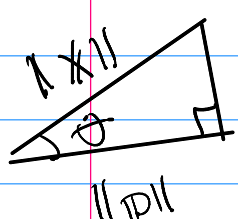



(i) Need to know p's length

(ii) Need same direction as y

easy: $u_1 = \dfrac{y}{\|y\|}$

so $p = (\text{length } p) \cdot u_1$




by trig $\cos\theta = \dfrac{\|p\|}{\|x\|}$

so $\|x\| \cos\theta = \|p\|$

but $\cos\theta = \dfrac{x^T y}{\|x\| \, \|y\|} \Rightarrow \|p\| = \dfrac{x^T y}{\|y\|}$

So length $\|p\| = \boxed{\|p\| = \dfrac{x^T y}{\|y\|}}$ Scalar proj. of x onto y

So $p = \|p\| \cdot u_1 = \dfrac{x^T y}{\|y\|} \cdot \dfrac{y}{\|y\|} = \dfrac{x^T y}{y^T y} \cdot y$

$$\boxed{\mathbb{P} = \frac{\mathbb{x}^T \mathbb{y}}{\mathbb{y}^T \mathbb{y}} \cdot \mathbb{y}}$$

Vector proj. of $\mathbb{x}$ onto $\mathbb{y}$