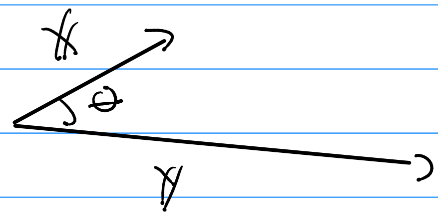
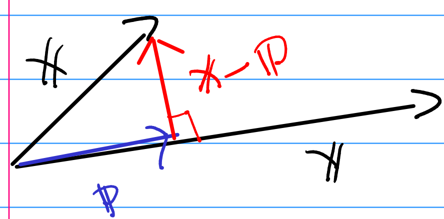
Math 511

$\mathbb{R}^n$ use $x^T y$ to study magnitude / related directions



(1) $\|x\| = (x^T x)^{1/2}$

(2) $\cos\theta = \dfrac{x^T y}{\|x\| \, \|y\|}$



$p = \dfrac{x^T y}{y^T y}\, y$ vector proj. of x onto y

$\|p\| = \dfrac{x^T y}{\|y\|} = \|x\| \cos\theta$ scalar proj. of x onto y

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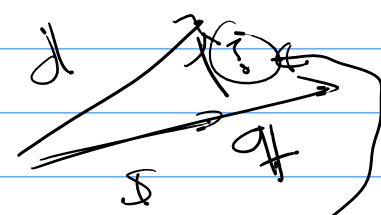
ex: find proj of d onto q, call it s

$s = \dfrac{d^T q}{q^T q}\, q$

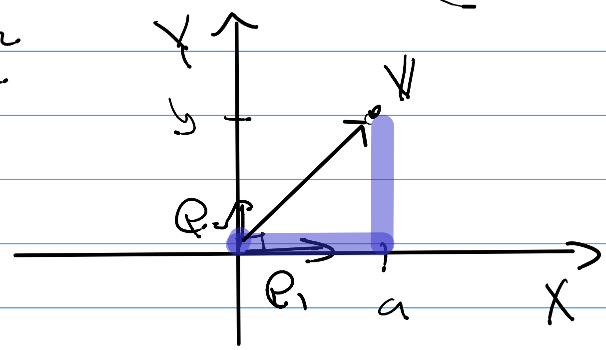


vector part of d orthogonal to s

$\boxed{d - s}$

$\boxed{5.2}$ Orthogonal Subspaces (goal: Fundamental Subspaces of $\mathbb{R}^n$)

Idea $\mathbb{R}^2$



$X = \text{span}(e_1)$

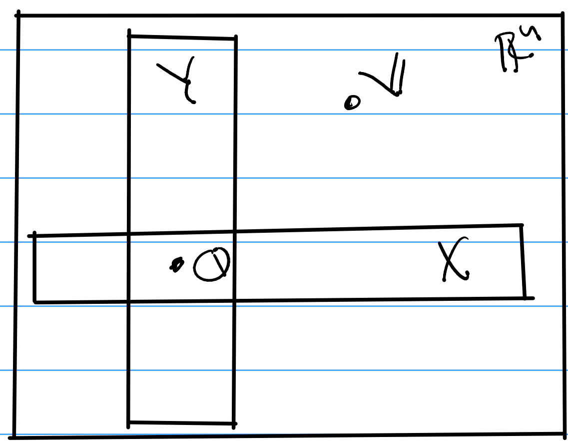
$Y = \text{span}(e_2)$ $\quad$ $y \in Y$

$v = \begin{bmatrix} a \\ b \end{bmatrix} = a e_1 + b e_2$

$x \in X$

$\mathbb{R}^n$

goal



want

① $X \perp Y$

② $v = x + y$ $\quad$ unique.

$x \in X$, $y \in Y$

Def X, Y are orthogonal subspaces of $\mathbb{R}^n$ (notation $X \perp Y$) if

for all $x \in X$, $y \in Y$, $\quad x^T y = 0$

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ex $\mathbb{R}^3$ basis $\{e_1, \boxed{e_2}, e_3\}$

i) $X = \text{span}(e_1)$, $\quad Y = \text{span}(e_3)$

$\to X \perp Y$ $\quad$ any $x \in X$ $\quad x = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

any $y \in Y$ $\quad y = b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$

and $x^T y = a\cdot 0 + 0\cdot 0 + 0\cdot b = 0$ orthogonal

$\therefore X \perp Y$

Def Y is a subspace of $\mathbb{R}^n$ then the set of all x in $\mathbb{R}^n$ such that $x^T y = 0$ for all $y \in Y$ is called Y's orthogonal complement

Notation: $Y^\perp = \{ x \in \mathbb{R}^n \mid x^T y = 0 \text{ for all } y \in Y \}$

ex $\mathbb{R}^3$

e_1, e_2, e_3 basis

$\boxed{X = \text{span}(e_1)}$

$X \perp \text{span}(e_2)$

$X \perp \text{span}(e_3)$

$X^\perp = \text{span}(e_2, e_3)$

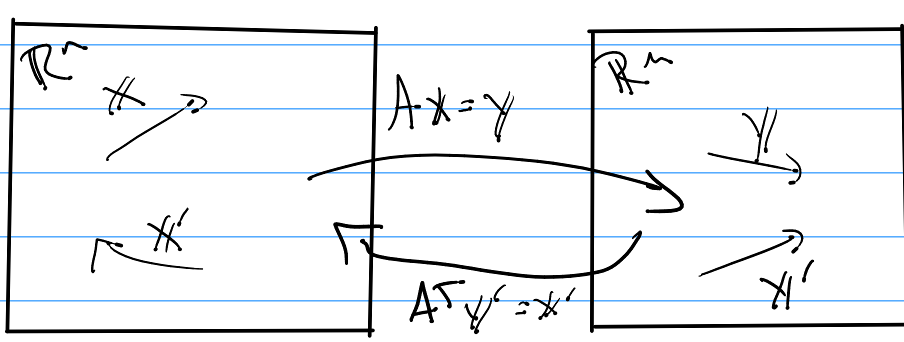
$X \perp \text{span}(e_2, e_3)$

Note: $Y \cap Y^\perp = \{0\}$ $\qquad A \cup B$

intersect (and) $\qquad$ union (or)

Now Consider A an $m \times n$ matrix (A^T is $n \times m$)

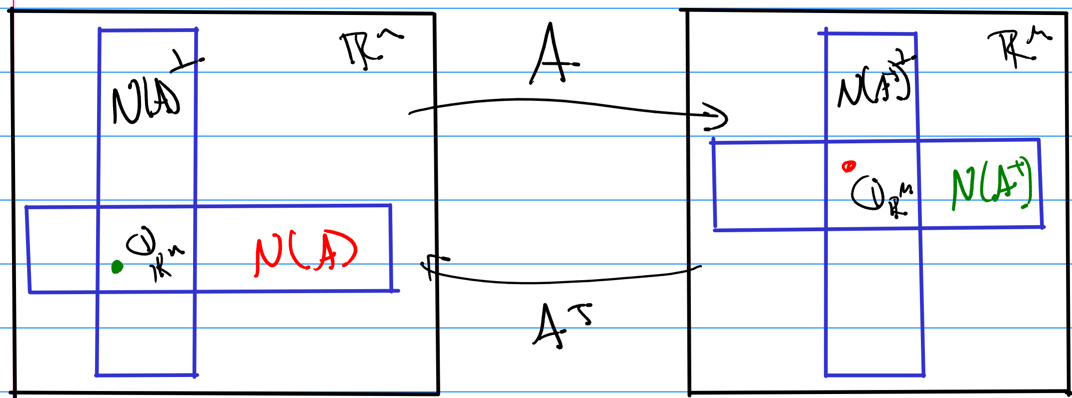
(Ax as a linear operator)



$A: \mathbb{R}^n \to \mathbb{R}^m$

$A^T: \mathbb{R}^m \to \mathbb{R}^n$

A, A^T are going to break $\mathbb{R}^n, \mathbb{R}^m$ into fund. subspaces



$\boxed{\text{Th}^m}$ Fundamental Subspaces th^m (for $A_{m \times n}$)

(1) $N(A)^{\perp} = R(A^T)$ where $R(A)$ is range of A & $R(A^T)$ is range of A^T

(2) $N(A^T)^{\perp} = R(A)$

(1a) $N(A) = R(A^T)^{\perp}$

(2a) $N(A^T) = R(A)^{\perp}$

Null Space:

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0_{\mathbb{R}^m} \}$$

$$N(A^T) = \{ y \in \mathbb{R}^m \mid A^T y = 0_{\mathbb{R}^n} \}$$

Range:

$$R(A) = \{ y \in \mathbb{R}^m \mid y = Ax \text{ for some } x \in \mathbb{R}^n \}$$

$$R(A^T) = \{ x \in \mathbb{R}^n \mid x = A^T y \text{ for some } y \in \mathbb{R}^m \}$$

Note: $R(A)$ is just col. space of A

$R(A^T)$ is just col. space of A^T

also: col space of A^T ← switch row A & col A

→ row space of A written as cols.

col space of A

⇒ row space of A^T written as cols.

together: ① range of A ⇒ col space of A ⇒ row space of A^T as col

② range of A^T ⇒ col space of A^T ⇒ row space of A as col.

Now: $A\vec{x} = \vec{0}$

as a system of eqn's ⇒

scalar prod of $\vec{x}$ and A's 1st row

$$\left\{\begin{array}{l} \boxed{x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} = 0} \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} = 0 \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \cdots + a_n a_{mn} = 0 \end{array}\right.$$

A^T's 1st col

b/c scalar prod is 0 we see $\vec{x}$ is orthogonal to each of A's rows, really each of A^T's col's.

$\therefore\ N(A) \perp$ col space(A^T)

and $N(A)^\perp = R(A^T)$

$\boxed{\text{Th}^m}$ Fund. Subspaces

$$N(A)^{\perp} = R(A^T) \quad \text{or} \quad N(A) = R(A^T)^{\perp}$$

$$N(A^T)^{\perp} = R(A) \quad \text{or} \quad N(A^T) = R(A)^{\perp}$$

$\boxed{\text{Th}^m}$ S is a subspace of $\mathbb{R}^n$. $\mathbb{R}^n$ has

basis $x_1, x_2, .., x_n$

If basis of S is $\{x_1, x_2, .., x_r\}$

$\dim(S) = r$

then $\dim(S^{\perp}) = n - r$

basis of $S^{\perp} = \{x_{r+1}, .., x_n\}$

$\dim(S) + \dim(S^{\perp}) = n$