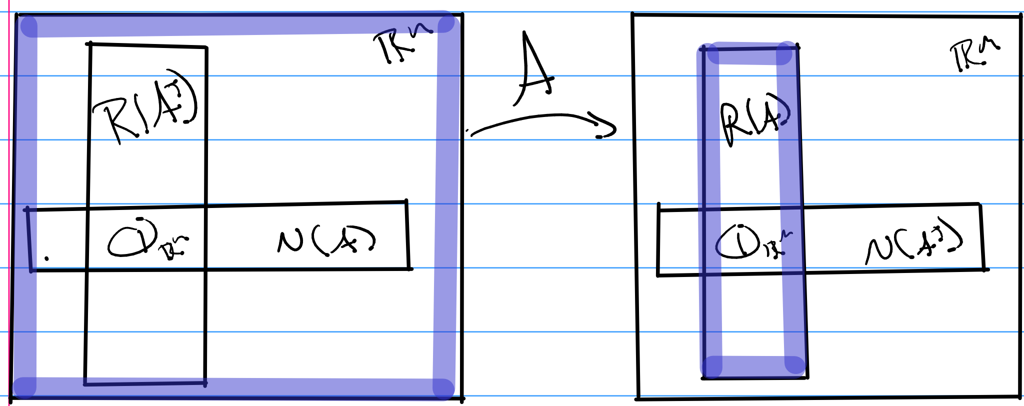
Math 511

$A_{m \times n}$, $A^T_{n \times m}$ $\qquad A\mathbf{x}$



think of $R(A^T)$ (to focus on A) as A's row space as col's

Ex $\quad A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & -4 & 1 & 4 \\ 3 & -6 & 2 & -1 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} U = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R(A)$ = col space of A = $\text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right)$

$R(A^T)$ = row space of A = $\text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right)$

(as) cols

$N(A) \rightarrow \left[U \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right] \rightarrow \mathbf{x} = \begin{matrix} 2\alpha - \beta \\ \alpha \\ -2\beta \\ \beta \end{matrix}$

$N(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$

Note! $\quad N(A)^{\perp} = R(A^T) \quad$ check

$$N(A) = \text{Span}\left(\underset{v_1}{\begin{bmatrix}2\\1\\0\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-1\\0\\-2\\1\end{bmatrix}}\right) \qquad R(A^T) = \text{Span}\left(\underset{v_3}{\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}0\\0\\1\\2\end{bmatrix}}\right)$$

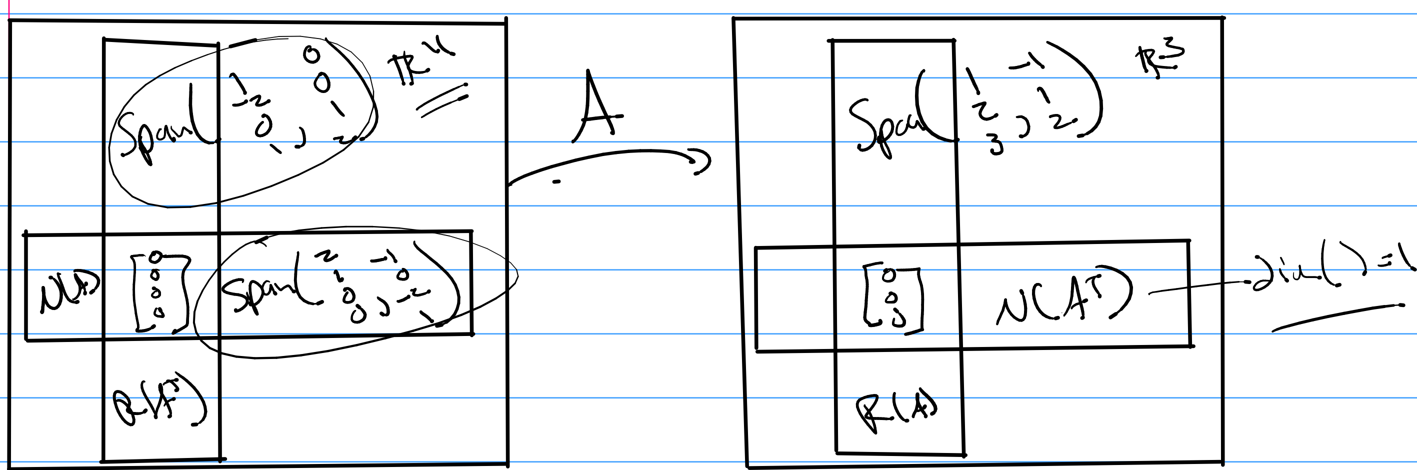
if $N(A)^\perp = R(A^T)$ check $v_1^T v_3 = 0$ $\quad v_1^T v_4 = 0$

$v_2^T v_3 = 0$ $\quad v_2^T v_4 = 0$

a basis for $\mathbb{R}^4$ could be $\begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}$

Summary

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & -4 & 1 & 4 \\ 3 & -6 & 2 & -7 \end{bmatrix}$$



Def: U, V are subspaces of W and every $w \in W$ can be uniq. written as $u + v$, $u \in U$, $v \in V$ then we say W is a direct sum of U, V

Notation: $U \oplus V = W$

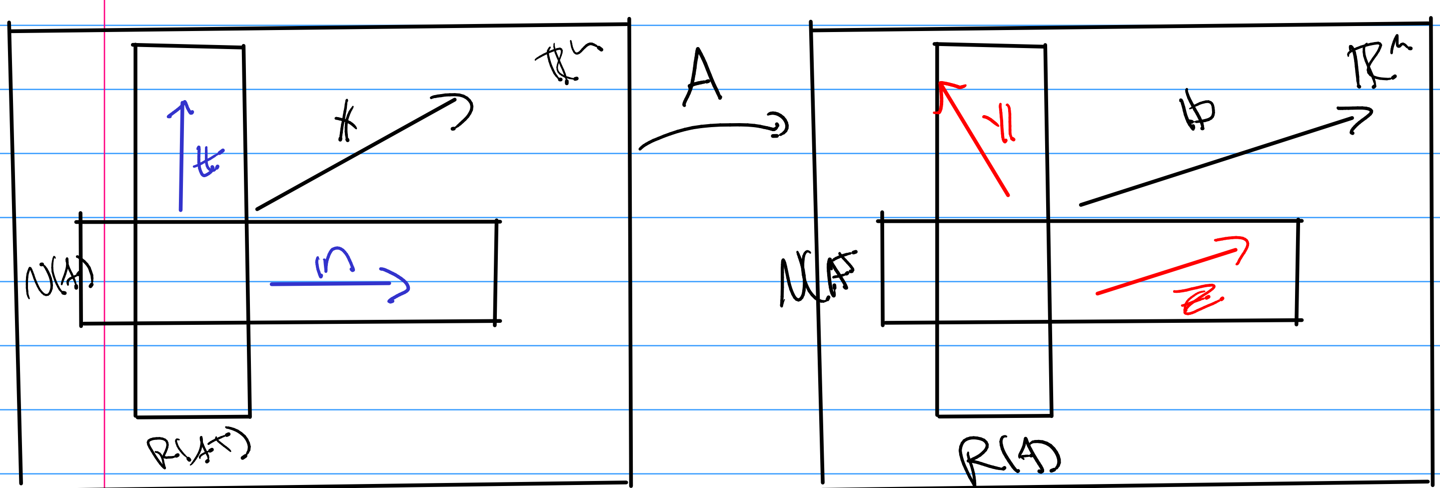
Thm $\mathbb{R}^n = S \oplus S^\perp$ for any subspace S

(*) **Thm** $(S^\perp)^\perp = S$

→ Fund. Subspace Thm

$N(A)^\perp = R(A^T)$ $\quad N(A) = R(A^T)^\perp$

$N(A^T)^\perp = R(A)$ $\quad N(A^T) = R(A)^\perp$



Uniq $x = t + n$ $\qquad$ **Uniq** $b = y + z$

Corollary if A is $m \times n$ and $b = \mathbb{R}^m$

Consider $Ax = b$

either ① $\exists x \in \mathbb{R}^n$ so that $Ax = b$ (b is in $R(A)$)

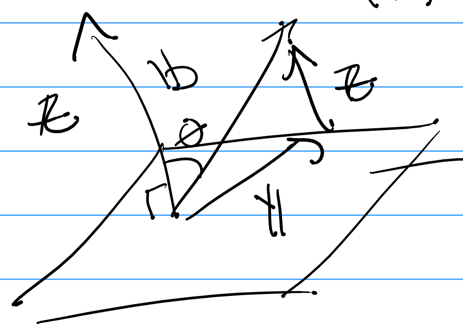
(says $Ax = b$ has a soln)

(used to say if $\mathbb{b} \notin R(A)$ that $A\mathbb{x} = \mathbb{b}$ has no solution, inconsist)

② $\exists\ \mathbb{z} \in \mathbb{R}^m$ such that

i) $A^T \mathbb{z} = \mathbb{0}_{\mathbb{R}^n}$ $\quad (\mathbb{z} \in N(A^T))$
$\qquad\qquad\qquad\qquad\qquad (\mathbb{z} \in R(A)^\perp)$

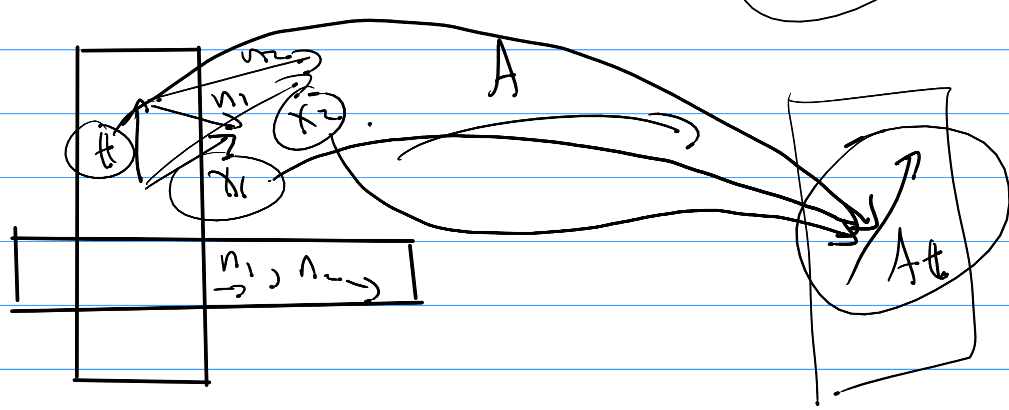
ii) $\mathbb{z}^T \mathbb{b} \neq 0$



(continued in 5.3)

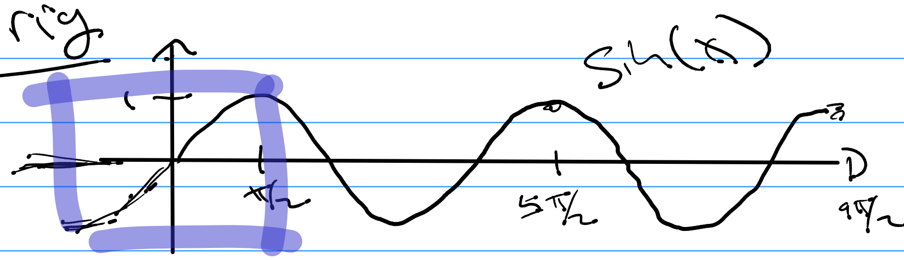
Notice: if $\mathbb{x} = \mathbb{t} + \mathbb{n}$, $\mathbb{t} \in R(A^T)$, $\mathbb{n} \in N(A)$
(unique)

then $A\mathbb{x} = A(\mathbb{t} + \mathbb{n})$
$= A\mathbb{t} + A\mathbb{n} = A\mathbb{t}$ $\quad (A\mathbb{n} = \mathbb{0})$



obvious A is not a one-to-one corresp.

ex) Trig



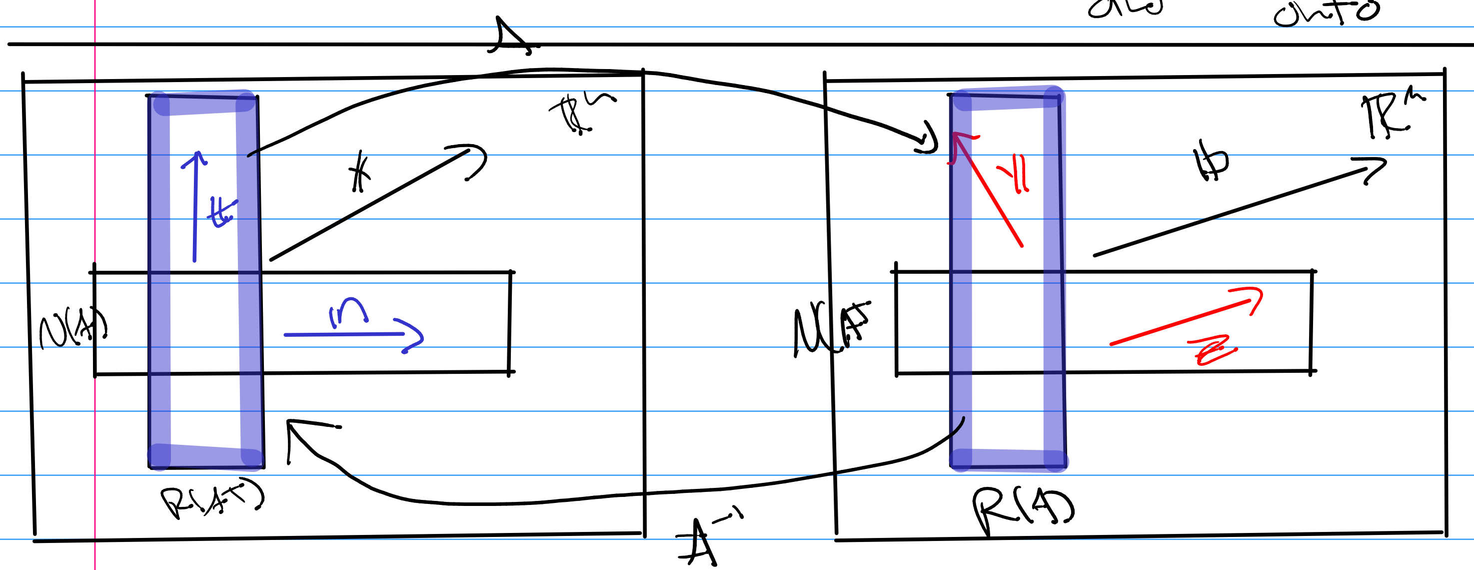
$x = t + n$ unique.

$$R(A) = \{ Ax \mid x \in \mathbb{R}^n \} = \{ At \mid t \in R(A^T) \}$$

<u>but</u> $Ax = A(t+n) = At$

so if you restrict the domain to just $R(A^T)$

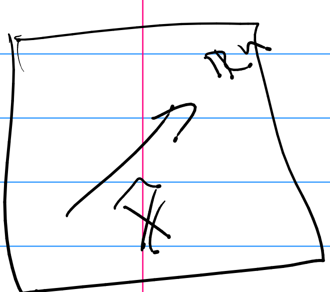
$A : R(A^T) \to R(A)$ is one-to-one and onto

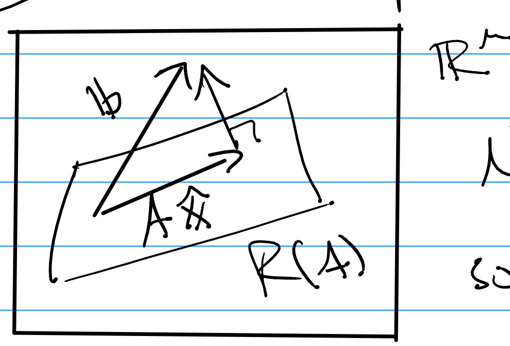


back to $Ax = b$ (Solve?)

(1) Yes if $b \in R(A)$

(2) No if $b \notin R(A)$ — no soln for $Ax = b$





<u>New problem</u>: Find $\hat{x} \in \mathbb{R}^n$ so that $A\hat{x}$ is <u>close</u> to b

close: minimize $\| A\hat{x} - b \|$

wr: $\| v \| = \sqrt{v^T v}$

Know: if you minimize c same as minimize c^2

$\Rightarrow \| v \|^2 = v^T v$

New Problem

$Ax = b$ has no soln. b/c $b \notin R(A)$

so **Solve** by minimize $\| Ax - b \|^2$

ans to this, $\hat{x}$, is called the least squared soln to $Ax = b$

real prob is least squares prob for $Ax = b$.

Next time

(to "solve" $Ax = b$ (has no soln)

is to solve $A^T A x = A^T b$)