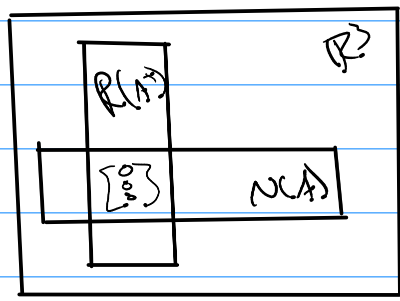
Math 511

Q's / (5.2 #6)

$(3,1,2)$ in rowspace

$\begin{bmatrix}2\\1\\1\end{bmatrix}$ in nullspace

? for this to be true?



$\xrightarrow{A}$

$N(A) \perp R(A^T)$

$\overset{\text{in}}{\begin{bmatrix}2\\1\\1\end{bmatrix}} = x \qquad \overset{\text{in}}{\begin{bmatrix}3\\1\\2\end{bmatrix}} = y$

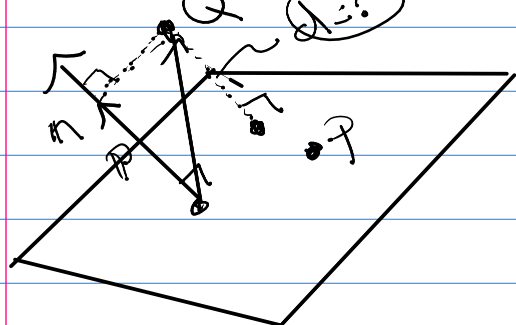
$x^T y = 9 \neq 0$

No!

5.1 #11 distance from $(2,1,-2)$

to $6(x-1)+2(y-3)+3(z+4)=0$ (plane)

d = ?



$P = (1,3,-4)$

$n = \begin{bmatrix}6\\2\\3\end{bmatrix}$

to solve: $\overrightarrow{PQ}$ proj onto n

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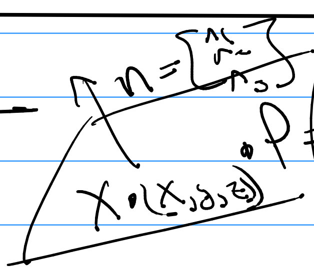
eqn of plane



know $n \perp \overrightarrow{XP}$

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$$

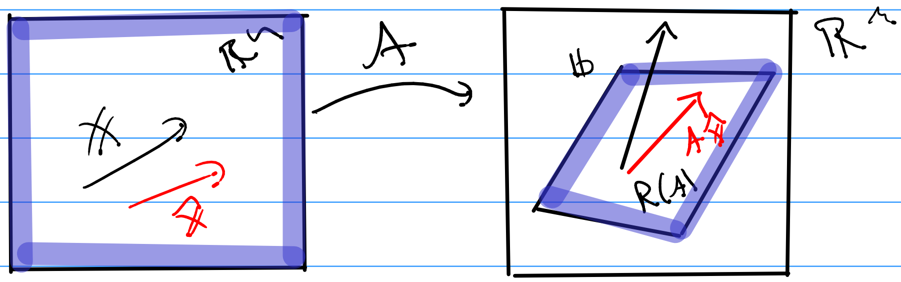
$$n^T \begin{bmatrix}x-a\\y-b\\z-c\end{bmatrix} = 0$$

$\boxed{5.3}$ Solve $A\mathbf{x} = \mathbf{b}$ but $\mathbf{b} \notin R(A)$

→ no solution! (inconsistent)

Study: if $A_{m\times n}$, $m > n$ (overdet system)

→ $\text{rank}(A) = n$ (no free variables)



$m > n$

Solve $A\mathbf{x} = \mathbf{b}$

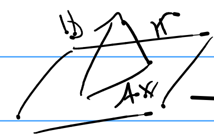
but $\mathbf{b} \notin R(A)$

instead of just saying "no soln"

→ Make a new problem

Find $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is close to $\mathbf{b}$

Least squares problem



given $A\mathbf{x} = \mathbf{b}$, let $\mathbf{r}(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$

Minimize $\|\mathbf{r}(\mathbf{x})\|^2 = \|\mathbf{b} - A\mathbf{x}\|^2$

for all $\mathbf{x} \in \mathbb{R}^n$

Soln: $\hat{\mathbf{x}}$ is called the least squares soln.

$\hat{x}$

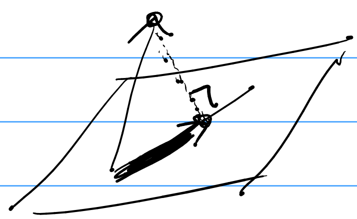
Exist?

thm: given S a subspace of $\mathbb{R}^n$, for any $\mathbf{b} \in \mathbb{R}^n$

there is a uniq. $\mathbf{p} \in S$ that is closest to $\mathbf{b}$

and $\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{y}\|$ for any $\mathbf{y} \neq \mathbf{p}, \mathbf{y} \in S$

iff $(\mathbf{b} - \mathbf{p}) \in S^\perp$



For our problem: the subspace $S = R(A)$

by thm is a $\mathbf{p} \in R(A)$ closest to $\mathbf{b}$

$\|\mathbf{b} - \mathbf{p}\|^2$ is a min

and $\boxed{\mathbf{b} - \mathbf{p} \in R(A)^\perp = N(A^T)}$

So there does exist an $\hat{x}$ so that

$$\boxed{A\hat{x} = \mathbf{p}}$$

so

$\rightarrow$ $\mathbf{b} - A\hat{x} \in N(A^T)$

for soln $\hat{x}$

being in $N(A^T)$ says $A^T(b - A\hat{x}) = 0$

$\rightarrow \quad A^T b - A^T A \hat{x} = 0$

says

$$\boxed{\begin{array}{c} A^T A \hat{x} = A^T b \\ \hat{x} \text{ is our soln (least-squares)} \end{array}}$$

So least square prob is $\boxed{\text{now}}$

$\boxed{\text{Solve}}$ $\boxed{A^T A}\, x = \boxed{A^T b}$

$M x = c$

$\rightarrow$ means: $A^T A$ better be invertable

$\rightarrow$ non singular

$\rightarrow$ $A^T A x = 0$ has only trivial soln

Finding Soln

$\boxed{\text{Th}^m}$ $A_{m \times n}$, $m \geq n$, $\text{rank}(A) = n$

$A^T A x = A^T b$ has a uniq. soln and

that soln is the least squares prob. soln

$\boxed{\text{so} \rightarrow}$ $\hat{x} = (A^T A)^{-1} (A^T b)$

$p = A\hat{x} = A(A^T A)^{-1}(A^T b)$

Call $A(A^TA)^{-1}A^T = P$

$\mathbb{P} = P\mathbb{b}$ (projection matrix for $\mathbb{b}$ onto $R(A)$)

Why? $A^TA\mathbf{x} = A^T\mathbb{b}$ has a soln if

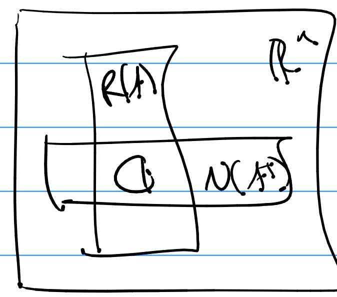
$A^TA\mathbf{x} = 0$ has <u>only</u> trivial soln.

assume $A^TA\mathbf{x} = 0 \rightarrow$ show … $\rightarrow$ $\mathbf{x} = 0$
(work)

$A^T(A\mathbf{x}) = 0$

<u>says</u> $A\mathbf{x} \in N(A^T)$

but obviously $A\mathbf{x} \in R(A)$

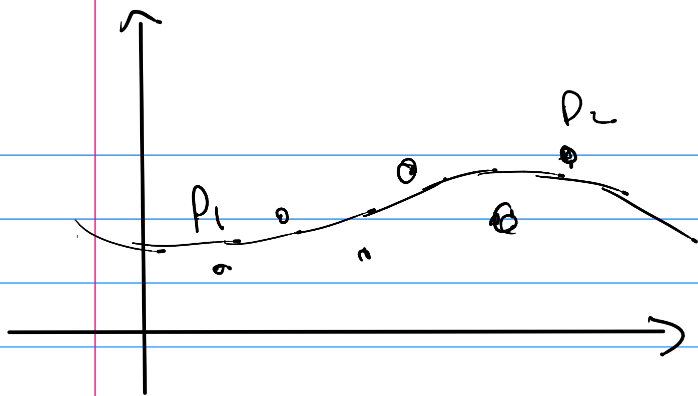


$\rightarrow$ $\boxed{A\mathbf{x} = 0}$ is a homogeneous system.

b/c <u>rank(A) = n</u> (no free variables)

<u>only</u> trivial soln $\rightarrow$ $\mathbf{x} = 0$

$\blacksquare$



$p_1 = (1,1)$
$p_2 = (2,2)$
$p_3 = (3,1)$
$p_4 = (4,3)$
$p_5 = (5,2)$
$p_6 = (6,3)$

$$p(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$$

$$g(x) = a + bx + cx^2$$

$p_1 = (1,1)$
$p_2 = (2,2)$
$p_3 = (3,1)$
$p_4 = (4,3)$
$p_5 = (5,2)$
$p_6 = (6,3)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

overdet

$$A c = y$$

no sol (no parab. through those 6 points)

least squares prob.

$$A^T A c = A^T y$$

GLS

$$\boxed{c = (A^T A)^{-1} (A^T y)}$$