Math 511

Q's/ Orthogonal complement of a subspace...

given $\mathbb{R}^n$ → n - vectors for a basis.

→ given S a subspace, $\dim(S) = r$

find $S^\perp = \text{span}($ n-r vectors $)$
lin. ind., $\boxed{\perp}$ to S

$\dim(S) + \dim(S^\perp) = n$

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ch5 $\mathbb{R}^n$ is a vector space ($\mathbb{X}$ is a col. vector ∴ objects
$\mathbb{X}_1 + \mathbb{X}_2$, $\alpha \mathbb{X}_1$ } are defined
→ the 10 axioms hold)

defined $\mathbb{X}^T \mathbb{Y} = x_1 y_1 + \cdots + x_n y_n$

↳ magnitude (length)

$$\|\mathbb{X}\| = (\mathbb{X}^T \mathbb{X})^{1/2}$$

angles

$$\cos\theta = \frac{\mathbb{X}^T \mathbb{Y}}{\|\mathbb{X}\| \|\mathbb{Y}\|}$$

Q/ Same idea for $\boxed{\text{any}}$ vector space

Inner Product Spaces

$\langle \mathbb{X}, \mathbb{Y} \rangle$ is an $\boxed{\text{inner product}}$

Is a vector space and we add an operator $\langle \mathbb{X}, \mathbb{Y} \rangle$ takes $\mathbb{X}, \mathbb{Y}$ and returns a scalar

has 3 properties ① $\langle \mathbb{X}, \mathbb{X} \rangle \geq 0$, also only $\mathbb{X} = 0$ has $\langle \mathbb{X}, \mathbb{X} \rangle = 0$

② $\langle \mathbb{X}, \mathbb{Y} \rangle = \langle \mathbb{Y}, \mathbb{X} \rangle$

③ $\langle \alpha \mathbb{X} + \beta \mathbb{Y}, \mathbb{Z} \rangle = \alpha \langle \mathbb{X}, \mathbb{Z} \rangle + \beta \langle \mathbb{Y}, \mathbb{Z} \rangle$

Note any inner product $\langle X, Y \rangle$ on a vector space creates an inner product space.

ex ① $\mathbb{R}^n$ with $\langle X, Y \rangle = \boxed{X^T Y}$

b/c this satisfies the 3 properties

$\rightarrow$ $\mathbb{R}^n$ with $X^T Y$ is an inner product space

② $\mathbb{R}^n$ and define $\langle X, Y \rangle$ by

① consider $w_1, w_2, \dots, w_n$ all pos numbers

② $\langle X, Y \rangle = w_1 x_1 y_1 + w_2 x_2 y_2 + \dots + w_n x_n y_n$

check: Does this meet

i) $\langle X, X \rangle = w_1 x_1^2 + w_2 x_2^2 + \dots + w_n x_n^2 \geq 0$

and only $= 0$ if $x_i = 0$

ii) $\langle X, Y \rangle = \langle Y, X \rangle$ true

iii) $\langle \alpha X + \beta Y, Z \rangle = \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle$

do the algebra... true

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typical inner products

① $\mathbb{R}^{m \times n}$ with $\langle A, B \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} b_{ij}$

ex $\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = -1 + 0 + 6 + 4 + 5 + 6$

$= 20$

above example

$$\| \mathbb{V} \| = \langle \mathbb{V}, \mathbb{V} \rangle^{1/2}$$

$$\left\| \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \right\| = \left(8^{1/2} \right) = \sqrt{8} = 2\sqrt{2}$$

$C[a,b]$

① inner product $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

② weighted $w(x) > 0$ on $[a,b]$

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$$

P_n

① inner product i) choose $x_1, x_2, \dots, x_n$ values

ii) $\langle p, q \rangle = p(x_1) q(x_1) + p(x_2) q(x_2) + \dots + p(x_n) q(x_n)$

② weighted i) choose $x_1, \dots, x_n$
have $w(x) > 0$

ii) $\langle p, q \rangle = w(x_1) p(x_1) q(x_1) + \dots + w(x_n) p(x_n) q(x_n)$

For an Inner Product space we can...

① $\| \mathbb{V} \| = \left(\langle \mathbb{V}, \mathbb{V} \rangle \right)^{1/2}$

② $| \langle \mathbb{X}, \mathbb{Y} \rangle | \le \| \mathbb{X} \| \, \| \mathbb{Y} \|$

$$\cos \theta = \frac{\langle \mathbb{X}, \mathbb{Y} \rangle}{\| \mathbb{X} \| \, \| \mathbb{Y} \|}$$

③ $\mathbb{X} \perp \mathbb{Y}$ when $\langle \mathbb{X}, \mathbb{Y} \rangle = 0$

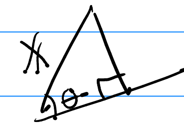
Pythagorean law

(4) if $\langle x, y\rangle = 0$ $\quad$ (so $x \perp y$)

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

(5) **Projections:**

x onto y $\qquad p = \dfrac{\langle x, y\rangle}{\langle y, y\rangle}\, y$



$$\| p \| = \frac{\langle x, y\rangle}{\| y \|} \qquad \left(\text{from } \cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \right)$$

(Ex) $C[-1,1]$ choose $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

consider: $f_1(x) = 1$

$f_2(x) = 1 + x$

$f_3(x) = x + x^3$

$$\| f_1 \| = \left(\langle f_1, f_1\rangle\right)^{1/2} = \left(\int_{-1}^{1} (1)^2\,dx\right)^{1/2} = \sqrt{2}$$

$$\| f_3 \| = \left(\langle f_3, f_3\rangle\right)^{1/2} = \left(\int_{-1}^{1} (x + x^3)^2\,dx\right)^{1/2}$$

$$= \left(\int_{-1}^{1} \left(x^2 + 2x^4 + x^6\right)dx\right)^{1/2}$$

$$= \left(2\left[\frac{1}{3} + \frac{2}{5} + \frac{1}{7}\right]\right)^{1/2}$$

unit function for $f_1 \rightarrow \dfrac{f_1}{\sqrt{2}} = \boxed{\dfrac{1}{\sqrt{2}}}$

angle from $f_1 = 1$ to $f_2 = 1+x$

$$\cos\theta = \frac{\langle f_1, f_2 \rangle}{\|f_1\| \, \|f_2\|}$$

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} (1)(1+x)\,dx = \int_{-1}^{1} (1+x)\,dx$$

$$= x + \cancel{\tfrac{1}{2}x^2}^{\,0} \Big|_{-1}^{1} = 2$$

$$\|f_1\| = \sqrt{2}$$

$$\|f_2\| = \left(\int_{-1}^{1} (1+x)^2\,dx\right)^{1/2} = \left(\int_{-1}^{1} (1+2x+x^2)\,dx\right)^{1/2}$$

$$= \left(x + x^2 + \tfrac{1}{3}x^3 \Big|_{-1}^{1}\right)^{1/2} = \left((1+\cancel{1}+\tfrac{1}{3}) - (-1+\cancel{1}-\tfrac{1}{3})\right)^{1/2}$$

$$= \left(2 + \tfrac{2}{3}\right)^{1/2} = \left(\tfrac{8}{3}\right)^{1/2} = \sqrt{\tfrac{8}{3}}$$

$$\cos\theta = \frac{2}{\sqrt{2}\sqrt{8/3}} = \frac{\cancel{2}}{\frac{\cancel{\sqrt{2}}}{1}\,\frac{2\cancel{\sqrt{2}}}{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\cos\theta = \sqrt{3}/2}$$

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app Show $f_1 = \sin(x)$ $f_2 = \cos(x)$

$f_1 \perp f_2$ on $\langle f, g \rangle = \int_{-\pi}^{\pi} \frac{1}{\pi} f(x) g(x)\,dx$

$C[-\pi, \pi]$

$\rightarrow$ $1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots$