Math 511

~~Q's~~ 5.4 #7 $C[-\pi,\pi]$ $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

$f_n(x) = \cos(nx)$ $g_n(x) = \sin(nx)$

$\{\cos(x), \cos(2x), \cos(3x), \dots, \sin(x), \sin(2x), \sin(3x), \dots\}$

check: ① $\langle \cos(mx), \sin(nx)\rangle \overset{?}{=} 0$

② $\langle \cos(m_1x), \cos(m_2x)\rangle \overset{?}{=} 0$ if $m_1 \neq m_2$

③ $\langle \sin(n_1x), \sin(n_2x)\rangle \overset{?}{=} 0$ if $n_1 \neq n_2$

① $\frac{1}{\pi}\int_{-\pi}^{\pi} \boxed{\cos(mx)\sin(nx)}\,dx = 0$

odd function

② $\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(m_1x)\cos(m_2x)\,dx = \frac{1}{\pi}\left[uv - \int v\,du\right]\Big|_{-\pi}^{\pi}$

idea: $u = \cos(m_1x) \rightarrow du = -m_1\sin(m_1x)$

$dv = \cos(m_2x) \rightarrow v = \frac{1}{m_2}\sin(m_2x)$

idea: $\cos(m_1x) = \cos(\underbrace{x + x + x + \dots + x}_{m_1})$

trig identities?

idea: table of integrals? $\boxed{\text{CAS?}}$

$\boxed{5.5}$ orthonormal sets.

V is a vector space with $\langle x, y \rangle$ inner product defined,

$\to$ Say $\boxed{V \text{ is an inner product space}}$

$\to$ $\{v_1, v_2, \ldots, v_k\}$ such that $\langle v_i, v_j \rangle = 0$ $i \neq j$

$v_i \perp v_j$ if $i \neq j$

call $v_1, v_2, \ldots, v_k$ an $\boxed{\text{orthogonal set}}$

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ex $C[-1, 1]$ $\langle f, g \rangle = \int_{-1}^{1} fg \, dx$

$p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = 3x^2 - 1$

$\langle p_1, p_2 \rangle = \int_{-1}^{1} x \, dx = 0$

$\langle p_1, p_3 \rangle = \int_{-1}^{1} (3x^2 - 1) dx = x^3 - x \Big|_{-1}^{1} = 0 - 0 = 0$

$\langle p_2, p_3 \rangle = \int_{-1}^{1} 3x^3 - x \, dx = 0$

So $1, x, 3x^2 - 1$ are an $\boxed{\text{orthogonal set}}$

$\boxed{\text{Th}^{\text{n}}}$ if $\{v_1, v_2, \ldots, v_k\}$ are an orthogonal set

$\to$ v_i are linearly independent

Pf idea: $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$

only trivial soln? $\Rightarrow$ ind.

$$\Rightarrow \langle v_i , c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \rangle = \langle v_i, 0 \rangle$$

$$c_1 \cancel{\langle v_i, v_1 \rangle} + c_2 \cancel{\langle v_i, v_2 \rangle} + \cdots + c_k \cancel{\langle v_i, v_k \rangle} = 0$$

$$c_i \underbrace{\boxed{\langle v_i, v_i \rangle}}_{\neq 0} = 0 \quad \rightarrow c_i = 0 \text{ any } i$$

Def $\{v_1, v_2, \ldots, v_k\}$ orthogonal set

and $\|v_i\| = 1 \rightarrow$ $\boxed{\text{orthonormal set}}$

ex $p_1 = 1 \quad p_2 = x \quad p_3 = 3x^2 - 1$

$$\|1\| = \left(\int_{-1}^{1} (1)^2 \, dx\right)^{1/2} = \sqrt{2}$$

$$\|x\| = \left(\int_{-1}^{1} x^2 \, dx\right)^{1/2} = \sqrt{2/3}$$

$$\|3x^2 - 1\| = \left(\int_{-1}^{1} (3x^2-1)^2 \, dx\right)^{1/2} = \left(\int_{-1}^{1} 9x^4 - 6x^2 + 1 \, dx\right)^{1/2}$$

$$= \left(2\left(\frac{9}{5} - 2 + 1\right)\right)^{1/2} = \sqrt{\frac{8}{5}}$$

$$\boxed{\text{so } p_1 = \frac{1}{\sqrt{2}}, \ p_2 = \sqrt{3/2}\, x, \ p_3 = \sqrt{5/8}\,(3x^2 - 1)}$$

orthonormal set

Given $u_1, u_2, \dots, u_k$ an orthonormal set

$\boxed{\text{then}} \Rightarrow$ basis for $\text{span}(u_1, u_2, \dots, u_k)$

call $U = \{u_1, u_2 \dots u_k\}$ an orthonormal basis

$\boxed{\text{th}^{m}}$ $U = [u_1, u_2 \dots u_k]$ is an orthonormal basis

of inner product space V

if $v \in V$, $v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$

$= [c]_U$

then $c_i = \langle v, u_i \rangle$

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ex $p_1 = \frac{1}{\sqrt{2}}$, $p_2 = \sqrt{\frac{3}{2}}\, x$, $p_3 = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$

$f(x) = 1 + x + x^2 = c_1 p_1 + c_2 p_2 + c_3 p_3$

$c_1 = \langle 1 + x + x^2, \frac{1}{\sqrt{2}} \rangle = \sim$

$c_2 = \langle 1 + x + x^2, \sqrt{\frac{3}{2}}\, x \rangle = \sim$

$c_3 = \langle 1 + x + x^2, \sqrt{\frac{5}{8}}\,(3x^2 - 1) \rangle = \sim$

$g(x) = \pi - ex + x^2 = [\quad]_P$

Why use $\mathcal{U}$ an orthonormal basis?

$\boxed{\text{Corollary}}$ $\quad \mathbf{v}_1 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_k\mathbf{u}_k = [\mathbf{a}]_u$

$\mathbf{v}_2 = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \dots + b_k\mathbf{u}_k = [\mathbf{b}]_u$

$\to \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = a_1b_1 + a_2b_2 + \dots + a_kb_k$

$= [\mathbf{a}]_u^T [\mathbf{b}]_u$

$\to \|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = a_1^2 + a_2^2 + \dots + a_k^2$

$= [\mathbf{a}]_u^T [\mathbf{a}]_u$

$\mathbb{R}^n$ $\quad Q = [\mathbf{q}_1, \mathbf{q}_2 \dots \mathbf{q}_n]$ has $\mathbf{q}_i$ are an orthonormal set

$\to$ call Q an orthogonal matrix

$\mathbf{q}_i^T \mathbf{q}_j = 0$ if $i \neq j$

$Q^T Q$

$$= \underset{n\times 1}{\begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix}} \underset{1\times n}{[\mathbf{q}_1, \mathbf{q}_2 \dots \mathbf{q}_n]} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 \dots & 0 \\ 0 & & \ddots & 1 \end{bmatrix}$$

$\mathbf{q}_i^T \mathbf{q}_i = 1$

So $Q^T Q = I$

So $Q^{-1} = Q^T$.

Properties Q is an orthogonal matrix

1. q_i are an orthonormal set
2. $Q^TQ = I$
3. $Q^{-1} = Q^T$
4. $\langle Q\mathbf{x}, Q\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$
5. Def: p-norm lengths

$$\|\mathbf{x}\| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

$$\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

so $\langle \mathbf{x}, \mathbf{x} \rangle^{1/2} = \|\mathbf{x}\|_2$

$$\boxed{\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2}$$

Least Squares Problem $A\mathbf{x} = \mathbf{b}$ overdet

rank$(A) = n$

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

A is orthogonal $\rightarrow A^TA = I$

least squares soln $\boxed{\hat{\mathbf{x}} = A^T\mathbf{b}}$