# Math 511

~~Q's~~ 5.9 #13 — $V$ is a vector space

inner prod.

a) add operator $\langle x, y \rangle$ that satisfies the 3 prop. in the text $\to$ inner product space

(2) b) add unary operator $\|x\| =$ scalar that satisfies the 3 properties..

**Norm**

1. $\|x\| \geq 0$, $\|x\| = 0$ on $\|0\| = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

$\to$ call $V$ with $\|x\|$ a $\boxed{\text{normed space}}$

ex $\boxed{\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|}$ show all 3 hold!

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unit vector given $v$

$$\text{unit vector } u = \frac{1}{\|v\|} v$$

inner prod space: $\|v\| = (\langle v, v \rangle)^{1/2}$

$$u = \frac{1}{\sqrt{\langle v, v \rangle}} v$$

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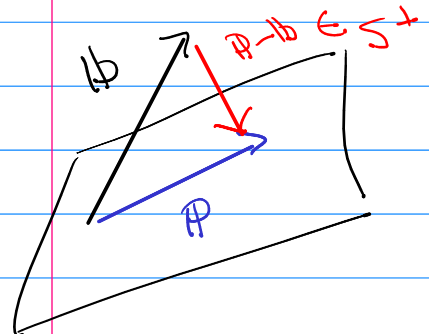
# Orthonormal Basis

least squares problem: $\hat{x} = (A^T A)^{-1} A^T b$

$p \in R(A)$ $\quad p = \underbrace{A(A^T A)^{-1} A^T}_{P}\, b$

So $P$ maps $b$ onto $R(A)$ given $p$ closest to $b$.

$p = P\,b$ ← proj. matrix of $b$ onto $R(A)$




$S$, subspace with orthonormal basis $u_1, u_2, \dots, u_k$

thm 5.5 $\quad p = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$

→ $c_i = \langle b, u_i \rangle$ then $p - b \in S^\perp$

ex $\quad e^x \approx c_1 p_1 + c_2 p_2 + c_3 p_3$

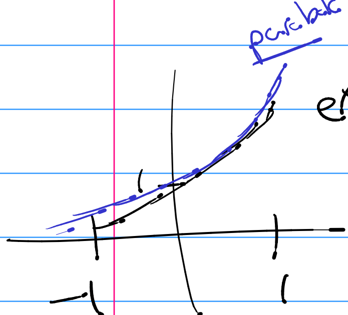
$p_1 = 1/\sqrt{2}$, $\quad p_2 = \sqrt{3/2}\, x$, $\quad p_3 = \sqrt{5/8}\,(3x^2 - 1)$

on $C[-1,1]$ $\quad \langle f, g \rangle = \int_{-1}^{1} f g \, dx$

$$c_1 = \langle e^x, 1/\sqrt{2} \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} e^x \, dx = \boxed{\phantom{xx}}$$

$$c_2 = \langle e^x, \sqrt{3/2}\, x \rangle = \int_{-1}^{1} \sqrt{3/2}\, x e^x \, dx = \boxed{\phantom{xx}}$$

$$c_3 = \langle e^x, \sqrt{5/8}(3x^2-1) \rangle = \int_{-1}^{1} \sqrt{5/8}(3x^2-1) e^x \, dx = \boxed{\phantom{xx}}$$




$$e^x \approx \underbrace{a + bx + cx^2}_{\text{are found by evaluating above}}$$

$$e^{100} = (e^{50})^2 = ((e^{25})^2)^2$$

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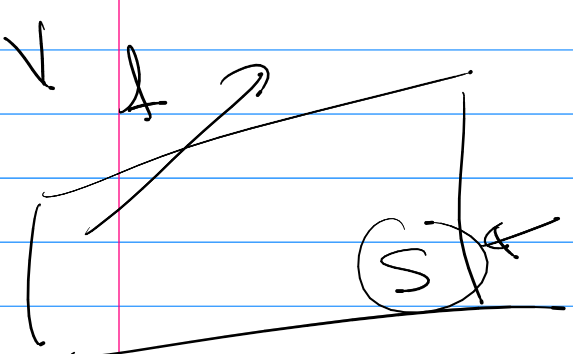
## Fourier Transform

$f(x)$ periodic function

by linear transform $\rightarrow$ $V$

$$C[-\pi, \pi]$$
$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, dx$$




$$\text{span}\left(\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx\right)$$

$c \sin x$ — $2\pi$ period

$d \sin nx$ — $\frac{2\pi}{n}$ period

$$f(x) \approx t_n(x) = a_0\left(\tfrac{1}{2}\right) + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

$$a_0 = \langle f, 1 \rangle$$

$$a_k = \langle f, \cos kx \rangle \qquad b_k = \langle f, \sin kx \rangle$$

5.6 $\mathbb{R}^n$ given $\boxed{\text{a basis}}$ ... orthonormal basis is better

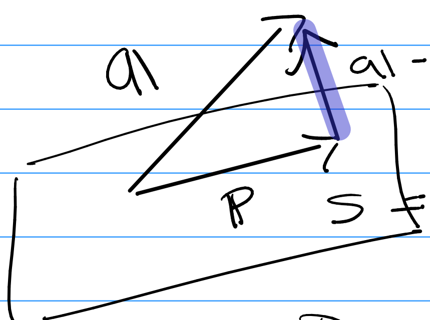
not orthonormal ?

given $A = [a_1, a_2 \dots a_n] \xrightarrow[\text{orthogonalization process}]{\text{Gram-Schmidt}} Q = [q_1, q_2 \dots q_n]$

not orthonormal is orthonormal

$\boxed{\text{idea}}$ (inductive process)

based on least squares approximation.




$S = \text{span}(q_1, q_2 \dots q_k)$

$$p = c_1 q_1 + c_2 q_2 + \dots + c_k q_k$$

$$p = \langle a_1, q_1 \rangle q_1 + \langle a_1, q_2 \rangle q_2 + \dots + \langle a_1, q_k \rangle q_k$$

Want: $a_1 - p \perp$ to $q_1, q_2, \dots, q_k$

Keep: $$q_{k+1} = \frac{1}{\|a_1 - p\|}(a_1 - p)$$

# Gram-Schmidt process | $A \to Q$

**Basis Step** $\quad q_1 = \frac{1}{\|a_1\|} a_1$

**Induction part** $\quad$ of $a_{k+1}$ (next $a$) $\quad$ know $q$'s

a) find projection $p$ onto $\text{span}(q_1, q_2, \dots, q_k)$

b) find difference $a_{k+1} - p$

c) keep $q_{k+1} = \frac{1}{\|a_{k+1} - p\|}(a_{k+1} - p)$

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ex $\quad a_1, a_2, a_3 \to q_1\ q_2\ q_3$

find $q_1$: $\quad q_1 = \frac{1}{\|a_1\|} a_1$ $\quad r_{11}$

find $q_2$:

a) $a_2$ on $\text{span}(q_1)$ (known) $\quad p = \langle a_2, q_1 \rangle q_1$ $\quad r_{12}$

b) $a_2 - p$

c) $q_2 = \frac{1}{\|a_2 - p\|}(a_2 - p)$ $\quad r_{22}$

find $q_3$:

a) $a_3$ onto $\text{span}(q_1, q_2)$ $\quad p = \langle a_3, q_1 \rangle q_1 + \langle a_3, q_2 \rangle q_2$ $\quad r_{13}$, $r_{23}$

b) $a_3 - p$

c) $q_3 = \frac{1}{\|a_3 - p\|}(a_3 - p)$ $\quad r_{33}$

etc

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$\underline{\underline{A = QR}}$$

by Gram-Schmidt we have factored $A$ into $A = QR$

$Q$: orthogonal matrix

$R$: upper triangular

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<u>why is QR interesting?</u> (Need $\underline{A = QR}$)

ex. Solve $Ax = b$

$\to QRx = b$ b/c $Q^{-1} = Q^T$

$$\boxed{Rx} = Q^T b$$

upper triangular

$\to$ Solve by back sub.