Math 511

Ch6 Eigen values and Eigenvectors (or Eigenspace) belonging to Eigen vectors

Previous

Markov Process   start x_0 basis

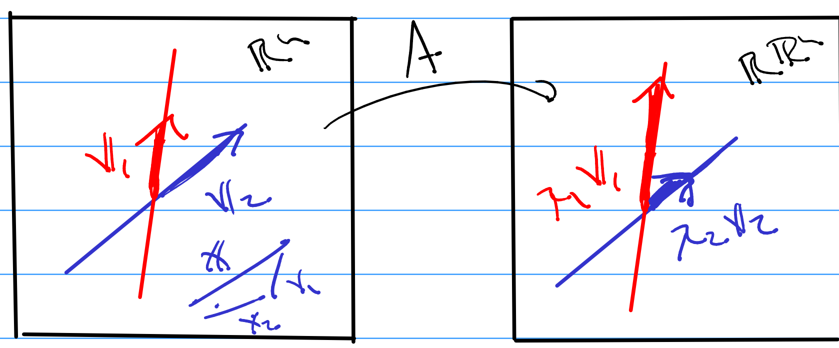
$x_1 = Ax_0, \quad x_2 = Ax_1, \dots \quad x_n = Ax_{n-1}$

or $\quad x_n = A^n x_0$

Markov Chain: $x_0, x_1, x_2, \dots$

there were directions such that $Ax_* = \lambda x_*$

$A_{n\times n}$



How to find λ's and x's belonging to the λ's?

eigen value   eigen vectors

Notation: ex ① $\lambda = 0.75$ $\quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ belongs to $\lambda = 0.75$

② eigenvalue is $\lambda = 0.75$
with eigen space $\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(3) eigenvalue $\lambda = 0.75$ with eigenvector

$$x = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

(4) eigenvalue is $\lambda = 0.75$

with eigenspace $= \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

Find: Solve $Ax = \lambda x$ for λ

$$Ax - \lambda x = 0 \qquad \text{b/c } x = Ix$$

$$\boxed{A}x - \boxed{\lambda I}x = 0$$

$$(A - \lambda I)x = 0 \quad \text{(Homogeneous system)}$$

Solve: λ, x that make this $\rightarrow$ 0 (Note: $x \neq 0$)

Means: find all non-trivial sol'ns to $(A - \lambda I)x = 0$

so

1. λ is an eigenvalue of A
2. $(A - \lambda I)x = 0$ has non-trivial sol'ns
3. $N(A - \lambda I) \neq \{0\}$
4. matrix $A - \lambda I$ is singular
5. $\det(A - \lambda I) = 0$

- Find λ? $\rightarrow$ find all λ such that $\det(A - \lambda I) = 0$
- for each λ_i, find x belonging to λ_i?

plug λ_i into $A - \lambda_i I$ solve $(A - \lambda_i I)x = 0$

$[A - \lambda_i I \mid 0]$ Solve. $N(A - \lambda_i I)$

(ex) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ eigenvalue / eigenvector

(1) $\boxed{\lambda's}$ Solve $\det(A - \lambda I) = 0$

b/c $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ $\boxed{\text{so}}$ $\boxed{\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix} = 0}$

Solve for λ

$\rightarrow (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{vmatrix} = 0$

$\rightarrow (1-\lambda)\left[(3-\lambda)(-1-\lambda) - 5\right] = 0$

$(\lambda-1)\left[(3-\lambda)(1+\lambda) + 5\right] = 0$

$(\lambda-1)(8 + 2\lambda - \lambda^2) = 0$

$(-1)(\lambda-1)(\lambda^2 - 2\lambda - 8) = 0$

$\boxed{(-1)(\lambda-1)(\lambda-4)(\lambda+2) = 0}$ characteristic eqn's

$\lambda_1 = 1 \quad \lambda_2 = 4 \quad \lambda_3 = -2$

$\boxed{\text{Eigenspaces}}$ (for each λ_i)

(1) $\lambda_1 = 1$ $\quad A - 1 \cdot I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{bmatrix}$

all x's such that $(A - 1I)x = 0$

$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1/2 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1/2 & 0 \\ 0 & 0 & -9/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 is free

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} x_1 &= \alpha \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \qquad X = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda = 1 \quad \text{eigenspace} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}$$

do same for λ_2, λ_3

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properties

① b/c characteristic eqn $\rightarrow$ characteristic poly $= 0$

$p(\lambda)$

$p(\lambda) = 0 \qquad \text{degree}(p(\lambda)) = n$

so $\lambda_1, \lambda_2, \ldots, \lambda_n$

ex $(\lambda - 2)(\lambda - 2)(\lambda + 3)(\lambda - 4) = 0$

$\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -3, \lambda_4 = 4$

ex $(\lambda + (3 - i))(\lambda + (3 + i)) = 0$

If $\lambda_i = a + bi$ is an eigen value with Z being it's eigen vector

then $\overline{\lambda_i} = a - bi$ is an eigen value as well with $\overline{Z}$ being it's eigen vector

ex $\lambda_1 = 1 + 2i$ with eigen vector $X = \begin{bmatrix} 1 + i \\ i \end{bmatrix} \rightarrow \lambda_2 = 1 - 2i$ with $X = \begin{bmatrix} 1 - i \\ -i \end{bmatrix}$

Note:

$$\boxed{\begin{aligned}\lambda_2 &= 1-2i \\ X &= \begin{bmatrix} 1-i \\ -i \end{bmatrix}\end{aligned}}$$

next person

$$\boxed{\begin{aligned}\lambda_2 &= 1-2i \\ X &= \begin{bmatrix} 1+i \\ 1 \end{bmatrix}\end{aligned}}$$

$$i\begin{bmatrix} 1-i \\ -i \end{bmatrix} = \begin{pmatrix} i-i^2 \\ -i^2 \end{pmatrix}$$

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properties

② if A is triangular $\rightarrow \lambda_i = a_{ii}$

ex
$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & -1 & 3 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda)(-1-\lambda)=0$$

$$\underline{\lambda=2} \quad \underline{\lambda=4} \quad \underline{\lambda=-1}$$

③ $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

④ trace of $A = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$