Math 511

Q5 5.5 #8 $C[-\pi,\pi]$ $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

Know: ① $\sin x, \cos x$ are orthonormal on $\uparrow$ inner prod space

② $\langle v_1, v_2\rangle$ given $v_1 = [a]_u$ $v_2 = [b]_u$

$\to = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ coord. in orthonormal basis

#8 $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (\underbrace{3\cos x + 2\sin x}_{\begin{bmatrix}3\\2\end{bmatrix}})(\underbrace{\cos x - \sin x}_{\begin{bmatrix}1\\-1\end{bmatrix}})\,dx$

$= 3\cdot 1 + 2\cdot(-1) = \boxed{1}$

5.3 #5

x	y
-1	0
0	1
1	3
2	9

fit linear function

$y = ax + b$

$$\begin{bmatrix} a(-1)+b = 0 \\ a(0)+b = 1 \\ a(1)+b = 3 \\ a(2)+b = 9 \end{bmatrix} \to \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A}\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{y}$$

"solve" $\begin{bmatrix} a \\ b\end{bmatrix} = (A^TA)^{-1}A^Ty$

Ch 6 Eigenvalues | Eigenvectors | Eigenspaces

given $A_{n \times n}$ ① λ_i $\det(A - \lambda I) = 0$

$= p(\lambda)$ characteristic poly

$Ax = \lambda x$

$(A - \lambda I)x = 0$

→ find $\lambda_1, \lambda_2, \ldots, \lambda_n$

② x_i Solve for each λ_i $[A - \lambda_i I \mid 0]$

6.3 Diagonalization. Eigenproblem: $Ax = \lambda x$

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$Ax_3 = \lambda_3 x_3$

$\vdots$

→ as block multiplication

$$A[x_1\ x_2 \cdots x_n] = [\lambda_1 x_1\ \lambda_2 x_2 \cdots \lambda_n x_n]$$

so
$$A[x_1\ x_2 \cdots x_n] = [x_1\ x_2 \cdots x_n]\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ \vdots & 0 & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

let $X = [x_1\ x_2 \cdots x_n]$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

→ $$\boxed{AX = XD}$$

Start with A → find λ_1, x_1 ; λ_2, x_2 ; .. ; λ_n, x_n

write $AX = XD$ $\quad X = [x_1\ x_2 \ldots x_n]$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Now if X^{-1} exists (x_i are linearly indep)

$$\boxed{\begin{array}{l} AX = XD \qquad \text{call } A \text{ diagonalizable} \\ \text{can be: } \quad A = XDX^{-1} \quad (\text{if not.. call } A \text{ defective.}) \\ \qquad D = X^{-1}AX \end{array}}$$

$\boxed{\text{consider}}$ A as a linear transform. → A_E is $L: \mathbb{R}^n \to \mathbb{R}^n$

X^{-1} exists → x_i are LI → x_i are a basis

So L in basis X

$$w_X = \boxed{X^{-1}AX}\, v_X$$

So $D = X^{-1}AX$ is the linear transform in basis of eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\boxed{A = XDX^{-1}}$$

x_i independent? $A_{n\times n}$

① find n-distinct $\lambda_i \to x_i$ are indep.

② what if λ_i has non-distinct values?

ex $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = -1,\ \lambda_4 = 2$

$\to$ find all x_1, x_2, x_3, x_4

$\boxed{\text{Now}}$ check ind of them. (use $\det(X)$)

Note: 3 "factors" of A

① $A = LU$

② $A = QR$

③ $AX = XD \to A = XDX^{-1}$

$D = X^{-1}AX$

Is $A = XDX^{-1}$ or $D = X^{-1}AX$ useful?

consider Markov process

$x_0,\ x_1 = Ax_0,\ x_2 = Ax_1 = A^2x_0, \ldots$

$x_n = A^n x_0$

Notice: $\begin{bmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{bmatrix} = B$, $B^2 = \begin{bmatrix} b_{11} & & 0 \\ & b_{22} & \\ 0 & & b_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & & 0 \\ & b_{22} & \\ 0 & & b_{nn} \end{bmatrix}$

$= \begin{bmatrix} b_{11}^2 & & 0 \\ & \ddots & \\ 0 & & b_{nn}^2 \end{bmatrix}$

$$B^k = \begin{bmatrix} b_{11}^k & & 0 \\ & \ddots & \\ 0 & & b_{nn}^k \end{bmatrix}$$

Now $A^k = (XDX^{-1})^k = (XD\underbrace{X^{-1})(X}_{I}DX^{-1}) \cdots (XDX^{-1})$

$$A^k = XD^kX^{-1} = X\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}X^{-1}$$

Matrix Ops $\quad A+B,\ AB,\ \alpha A,\ A^{-1},\ A^k$

Consider: $\exp(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

real numbers

Matrix $\quad \boxed{\exp(M) = e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots}$

(F) $\quad A = XDX^{-1}$

$$\exp(A) = e^A = I + A + \frac{1}{2!}A^2 + \dots$$

$$\exp(A) = \exp(XDX^{-1}) = \underbrace{I}_{XIX^{-1}} + XDX^{-1} + \underbrace{\frac{1}{2!}XD^2X^{-1}}_{X\frac{1}{2!}D^2X^{-1}} + \dots$$

$$\exp(A) = X\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots\right)X^{-1}$$

$$\exp(A) = X\begin{bmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\dots & & 0 \\ & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\dots & \\ 0 & \ddots & 1+\lambda_n+\frac{1}{2!}\lambda_n^2+\dots \end{bmatrix}X^{-1}$$

$$\exp(A) = X \begin{bmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \; e^{\lambda_n} \end{bmatrix} X^{-1} \qquad V_E$$

$$\underbrace{\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}}_{\text{call } e^{D}}$$

$$\boxed{e^{A} = X e^{D} X^{-1}}$$
