Math 511

6.2 (1e)

diff eq. $\boxed{y' = ay \quad @\ t=0 \quad y(0) = y_0}$

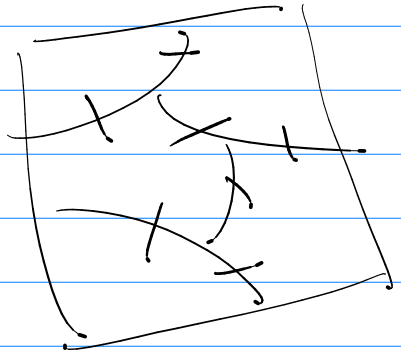
(1D)

guess $y = ce^{at}$ check $y' = cae^{at} = ay$

use $y(0) = y_0 \rightarrow y_0 = ce^0 = c$

so (sln) $\boxed{y = y_0 e^{at}}$

System of related rates



y_i is amount of (stuff) in tank i;

$\rightarrow y_i(t)$, $y_i'(t)$

$$y_1'(t) = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$$

$$y_2'(t) = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$$

$$\vdots$$

$$y_n'(t) = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$$

Now $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix} = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$ $Y' = \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}$

System: $Y' = AY$ initial value problem

$@\ t=0 \quad Y_0 = \begin{bmatrix} y_1(0) \\ \vdots \\ y_n(0) \end{bmatrix}$

Soln $Y = e^{tA} Y_0$

if $A = XDX^{-1}$

$$Y = X e^{tD} X^{-1} Y_0$$

$$\boxed{Y = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix} X^{-1} Y_0}$$

(6.2) type-soln guess $Y = \begin{bmatrix} x_1 e^{rt} \\ x_2 e^{rt} \\ \vdots \\ x_n e^{rt} \end{bmatrix} = e^{rt} \vec{x}$

check: $Y' = \frac{d}{dt}\left[e^{rt}\vec{x}\right] = re^{rt}\vec{x} = rY$

$\rightarrow AY = A e^{rt}\vec{x} = re^{rt}\vec{x} = rY = Y'$ ←

if $\vec{x}$ is an eigenvector with eigenvalue r then $A\vec{x} = r\vec{x}$

Soln for every $r_i, \vec{x}_i$ eigenvalue/vector

$$Y_i = e^{r_i t}\vec{x}_i$$

$$Y = c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n$$

$y_1' = 3y_1 - 2y_2$

$y_2' = 2y_1 + 3y_2$

$Y' = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} Y$ (A)

Y_0 Initial value

Soln $Y = X e^{tD} X^{-1} Y_0$

(Eigen work)

$A = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$

① r's $\det \begin{pmatrix} 3-r & -2 \\ 2 & 3-r \end{pmatrix} = 0$

$(3-r)^2 + 4 = 0$

$3 - r = \pm 2i$

$r = 3 \pm 2i$

② x's

i) $r = 3 - 2i$

$\begin{bmatrix} 3-(3-2i) & -2 \\ 2 & 3-(3-2i) \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$

$r_1 \cdot i + r_2 = \text{new } r_2$

$2i^2 - 2i$

-2

$\left[\begin{array}{cc|c} \boxed{2i} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

free

$x_2 = 2$

$x_2 = \alpha \quad x_1 = \frac{2\alpha}{2i}$

$x = \alpha \begin{bmatrix} \frac{1}{i} \\ 1 \end{bmatrix}$

$x = \alpha \begin{bmatrix} -i \\ 1 \end{bmatrix}$

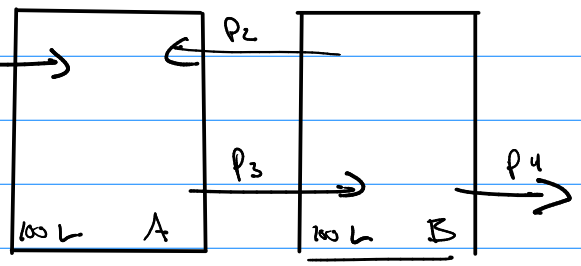
$r_1 = 3 - 2i \quad x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$r_2 = 3 + 2i \quad x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$Y = X \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix} X^{-1} Y_0$

$$Y = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(3-2i)t} & 0 \\ 0 & e^{(3+2i)t} \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1} Y_0$$

Ex



$y_i(t)$ = grams of salt in tank i

$$y_1(0) = \begin{bmatrix} 40 \\ 20 \end{bmatrix} = Y_0$$
$$y_2(0)$$

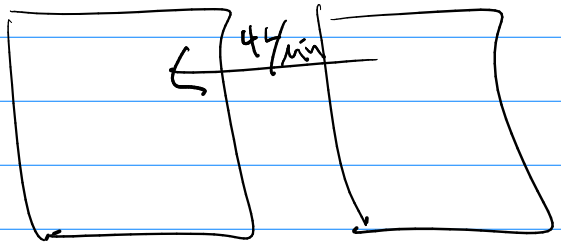
P_1 is 12 L/min of pure H_2O → 0

P_2 is 4 L/min of tank B's mixture → $\frac{y_2}{100}$

P_3 is 16 L/min of tank A's mixture → $y_1/100$

P_4 is 12 L/min of tank B's mixture → $y_1/100$

P2



$\frac{4\,L}{min}$ of B's mixture

$\frac{4\,L}{min}\left(\frac{y_2\ g\ of\ salt}{100\,L}\right)$

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tank A

$$y_1'(t) = (\text{Rate in}) - (\text{Rate out})\ \frac{4\,y_2}{100}\ \frac{g\ salt}{min}$$

$$y_1'(t) = \left(12\tfrac{L}{min} \cdot 0 + 4\tfrac{L}{min} \cdot \tfrac{y_2\,g}{100\,L}\right) - \left(16\tfrac{L}{min} \cdot \tfrac{y_1\,g}{100\,L}\right)$$

$$y_1'(t) = -\frac{16}{100} y_1 + \frac{4}{100} y_2$$

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tank B

$$y_2'(t) = (\text{Rate in}) - (\text{Rate out})$$

$$y_2'(t) = \left(16\tfrac{L}{min} \cdot \tfrac{y_1\,g}{100\,L}\right) - \left(\tfrac{4\,L}{min} \cdot \tfrac{y_2\,g}{100\,L} + \tfrac{12\,L}{min}\ \tfrac{y_2\,g}{100\,L}\right)$$

$$y_2'(t) = \frac{16}{100} y_1 - \frac{16}{100} y_2$$

Exam 3 — 11 probs @ 10pts each
100pts = 100%

5.1 — 1 prob (projectors)

5.2 — 1 prob — relationships b/w A, $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$
(draw them)

5.3 — 1 prob. — given data → fit it!

5.4 — 2 prob (projectors) $C[\Sigma, b]$, $\mathbb{R}^{m \times n}$, P_n

5.5 — 2 prob ① orthonormal?
② use coord. & orthonormal-basis.

5.6 — 1 prob — Gram-Schmidt.

(cont. next class)