Math 511

Final Exam

5 probs (variations from each exam) $\to$ 15 probs @ 10pts

$\to$ $\boxed{135 \text{ pts} = 100\%}$

$\boxed{\text{Exam 3}}$

(1) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = x \qquad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

x onto y

$$p = \frac{\langle x, y\rangle}{\langle y, y\rangle} y \qquad \|p\| = \frac{\langle x, y\rangle}{\|y\|} \qquad \cos\theta = \frac{\langle x, y\rangle}{\|x\|\|y\|}$$

a) $\langle x, y\rangle = 2$

b) $\langle y, y\rangle = 3$

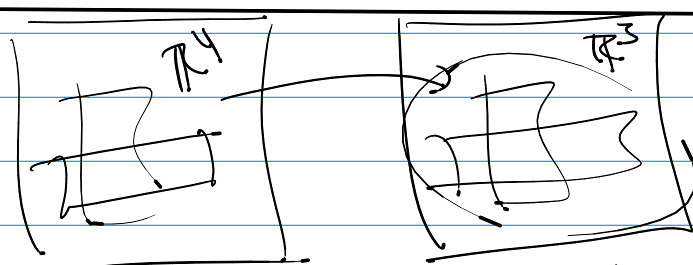
c) $\|y\| = \sqrt{3}$

$$p = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \|p\| = \frac{2}{\sqrt{3}}$$

Show $x - p \perp p \to [x - p]^T p = 0$

show

(2) $A_{3\times 4}$



$\boxed{\text{Can!}}$ $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \in N(A^T)$

and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in R(A)$?

check $\perp$? (No)

so $\boxed{\text{No}}$

$N(A)^\perp = R(A^T)$ row space of A as cols

$N(A^T)^\perp = R(A)$ col space of A

(3)
$$\begin{array}{cc} x & y \\ -1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 2 & 2 \end{array} \qquad \begin{Bmatrix} a \\ b \end{Bmatrix} = \left(\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right)$$

$$\{c\} = (A^T A)^{-1} (A^T \mathbb{y})$$

(4) $\mathbb{x}$ onto $\mathbb{y}$
$$\begin{array}{cc} 1 & 1 \\ 1 & x^2 \end{array} \qquad \mathbb{P} = \frac{\langle \mathbb{x}, \mathbb{y} \rangle}{\langle \mathbb{y}, \mathbb{y} \rangle} \mathbb{y}$$

$$p(x) = \frac{\cancel{2}\int_0^1 x^2 \, dx}{\cancel{2}\int_0^1 x^4 \, dx} (x^2) = \frac{\frac{1}{3}}{\frac{1}{5}} x^2 = \boxed{\frac{5}{3} x^2}$$

(5)
$$\cos\theta = \frac{\langle \mathbb{x}, \mathbb{y} \rangle}{\|\mathbb{x}\| \|\mathbb{y}\|} \qquad C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\cos\theta = \frac{1+0+2+0}{\sqrt{1^2+2^2+2^2+0^2}\sqrt{1^2+0^2+1^2+(-1)^2}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = \arccos\left(\frac{1}{3}\right)$$

(6) orthonormal
$$f_1 = \frac{1}{\sqrt{2}} \qquad f_2 = \sqrt{\frac{3}{2}}\, x$$

$$\mathbb{x} \perp \mathbb{y} \qquad \|\mathbb{x}\| = 1 \qquad \langle f_1, f_2 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \sqrt{\frac{3}{2}}\, x \, dx = \boxed{0}$$

$$\|f_1\| = \left(\int_{-1}^{1} \left(\frac{1}{\sqrt{2}}\right)^2 dx \right)^{1/2} = \sqrt{\frac{2}{2}\int_0^1 dx} = \sqrt{1} = \boxed{1}$$

g(c odd) funcn.

$$\|f_2\| = \left(\int_{-1}^{1} \frac{3}{2} x^2 \, dx \right)^{1/2} = \left(3\int_0^1 x^2 \, dx \right)^{1/2} = \sqrt{3\left(\frac{1}{3}\right)} = \boxed{1}$$

(7) $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f g \, dx$ $\qquad \sin x, \cos x$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (3\sin x - 2\cos x)(\cos x + 2\sin x)\, dx$$

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix}_u \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_u$$

$$= 6 - 2 = 4$$

(8) $\begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 2 \\ -2 & 2 & 1 \end{bmatrix} = A$ $\qquad Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$ $\quad R = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

q_1: $q_1 = \frac{1}{\|a_1\|} a_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

---

q_2: $P = \langle \underbrace{a_2, q_1}_{2} \rangle q_1 = \frac{2}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $\qquad a_2 - P = \begin{matrix} 4 \\ 3 \\ 2 \end{matrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$q_2 = \frac{1}{\|a_2 - p\|}(a_2 - P)$ $\qquad \begin{matrix} 4 \\ 3 \\ 2 \end{matrix} - \begin{matrix} 2/3 \\ 4/3 \\ -4/3 \end{matrix}$

$q_2 = \frac{1}{5}\left(\frac{5}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$ $\qquad a_2 - P = \begin{bmatrix} 10/3 \\ 5/3 \\ 10/3 \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

---

q_3: $P = \langle a_3, q_1 \rangle q_1 + \langle a_3, q_2 \rangle q_2 = \begin{bmatrix} 5/3 \\ 4/3 \\ 2/3 \end{bmatrix}$

$$1 \cdot q_1 \quad + \quad 2 \cdot q_2$$

$a_3 - P = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} - \begin{matrix} 5/3 \\ 4/3 \\ 2/3 \end{matrix} = \begin{matrix} -2/3 \\ 2/3 \\ 1/3 \end{matrix}$ $\qquad \|a_3 - p\| = 1$

$$\mathbb{X}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

---

(9) $A = \begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}$

$\boxed{\lambda_1 = 3}$

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \alpha \quad x_2 = 0$

$\boxed{\mathbb{X}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\boxed{\lambda_2 = -2}$

$$\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha$

$x_1 = -2/5\,\alpha$

$\boxed{\mathbb{X}_2 = \begin{bmatrix} -2/5 \\ 1 \end{bmatrix}}$

---

(10) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 + 4 = 0$

$1-\lambda = \pm 2i$

$\lambda = 1 \pm 2i$

$\lambda_1 = 1 - 2i$

$$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha$

$x_1 = 1/i\,\alpha = -i\alpha$

$\mathbb{X}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 = 1+2i \quad \mathbb{X}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

on final | 5 probs

1. projection of some sort
2. $A_{n \times n}$ explain properties of $N(A)$, $R(A^T)$ etc
3. Verify orthonormal stuff
4. Use coord. of orthonormal sets
5. Eigenvector / value problem (2×2)

Exam 1

1. 3×3 system (solve by gauss jordan)
2. ops matrix, vector
3. Matrix algebra (w/o actual numbers)

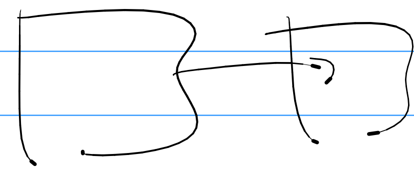
$$AX + B = X - 3C + BX$$

$$X = (\;?\;)$$

4. #s from text.
5. det(A)

Exam 2

1. Vector space?
2. are "they" linearly independent $\mathbb{R}^n$, P_n, $C[a,b]$
3. Make basis (pare down (extend))
4. like #8 A, u



5. $L: \overset{\mathbb{R}^n}{V} \to \overset{\mathbb{R}^n}{W}$

   L as a matrix