Math 321

$|\mathbb{Z}^+| = |\{1,2,3,\dots\}| = \aleph_0$ aleph null ("countable")

any S such that $|S| = \aleph_0$ or $|S| = n \in \mathbb{N}$
→ call S countable

ex) know: $|\text{evens}| = \aleph_0$ $|\text{odds}| = \aleph_0$

$|\mathbb{Z}| = \aleph_0$ $|\mathbb{Q}| = \aleph_0$ shown last class

Q) $|\mathbb{R}| = ?$

$\mathbb{R} = \mathbb{Q} \cup \text{Irrationals}$

$\sqrt{2}, \frac{\sqrt{2}}{2}, \dots$

$0.121121112111112\dots$

guess countable for $\mathbb{R}$?
means: bijection

0 1

$1 \longrightarrow r_1 = 0.d_{11}d_{12}d_{13}\dots$ just consider these reals
$2 \longrightarrow r_2 = 0.d_{21}d_{22}d_{23}\dots$
$3 \longrightarrow r_3 = 0.d_{31}d_{32}d_{33}\dots$
$4 \longrightarrow r_4 = 0.d_{41}d_{42}d_{43}\dots$
$\vdots$

$d_{ij} \in \{0,1,2,\dots,9\}$

the i^{th} real in my list
the j^{th} decimal of the real

and all reals from 0 to 1 are here in this list.

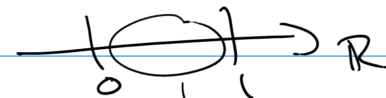
Prove $\mathbb{R}$ is uncountable

pf (use contradiction) → $\boxed{\text{Statement} \equiv \text{True}}$ ← want this but it's too hard

rather → $\boxed{\neg\,\text{statement} \equiv \text{False}}$ ← show this instead

So our goal is to show $\mathbb{R}$ is countable $\equiv$ False

assume $\mathbb{R}$ is countable. This means we can have $\mathbb{R}$ from 0 to 1 being countable



(Remember Countable means bijection from $\mathbb{Z}^+$) each $r = \boxed{0.}d_1 d_2 \dots$

$$1 \to r_1 = 0.d_{11}\, d_{12}\, d_{13} \dots$$
$$2 \to r_2 = 0.d_{21}\, d_{22}\, d_{23} \dots$$
$$3 \to r_3 = 0.d_{31}\, d_{32}\, d_{33} \dots$$
$$\vdots$$

and all reals from 0 to 1 are here.

<u>Note</u>: there is one problem for decimal numbers. Terminating decimals

$$0.123 = 0.122999999\dots$$

$$0.999\dots = 1.0$$

$$S = 0.999\dots \qquad 10S - S = 9$$
$$10S = 9.999\dots \qquad 9S = 9$$
$$S = 1$$

So for our list we will remove all $\overline{9}$ versions.

$\boxed{\text{Now}}$ our list of $r_1, r_2, r_3, \dots$ has all reals from 0 to 1 <u>and</u> each decimal representation is uniq.

We will now make a special real between zero and 1.

$$r^* = 0.d_1 d_2 d_3 \dots \qquad \text{and } d_i = \{1, 2, \dots, 8\}$$

and pick d_i such that ...

pick $d_1 \neq d_{11}$ $\left(\text{back} \quad d_1 = \begin{cases} 3 & \text{if } d_{11} \neq 3 \\ 4 & \text{if } d_{11} = 3 \end{cases} \right.$

by uniq rep. $r^* \neq r_1$

now

pick $d_2 \neq d_{22} \xrightarrow[\text{uniq}]{\text{by}} r^* \neq r_2$

$\boxed{\text{Continue}}$ $d_i \neq d_{ii} \longrightarrow \underline{r^* \neq r_i \text{ for all } i}$

list $r_1 = 0.d_{11}\ldots$ <u>all</u> reals are here

$r_2 = 0.d_{21}\ldots$ <u>and</u> r^* is not there. $\equiv$ False

$\vdots$ $r^* = 0.d_1 d_2 d_3 \ldots$

$\rightarrow |\mathbb{R}| \neq \aleph_0$ not countable

Call $\mathbb{R}$ uncountable

PES

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so $\mathbb{R} = \mathbb{Q} \cup \text{Irrational}$

$\mathbb{R}$ uncountable, $\mathbb{Q}$ countable, Irrational <u>uncountable</u>

so $|\mathbb{Z}^+| = \aleph_0 < |\mathbb{R}|$

$\boxed{\text{Interesting byproducts}}$

① Continuum Hypothesis

$|\mathbb{Z}^+| = \aleph_0$

$|P(\mathbb{Z}^+)| = 2^{|\mathbb{Z}^+|} = 2^{\aleph_0} = \aleph_1$

<u>and</u> $|\mathbb{R}| = \aleph_1$

$|\mathbb{Z}^+|$

$\downarrow$

$\aleph_0 < \aleph_1 < \aleph_2 < \dots$

transfinite cardinals

Computability $\to$ "you can write a program"