Math 243

Inverse $y = f(x)$ (invertable on its domain)

Invert → $\boxed{x = f(y)}$ → algebra? → $\boxed{y = f^{-1}(x)}$ $\overset{?}{=}$ expression of x's (if possible)

implicit representation of f^{-1}

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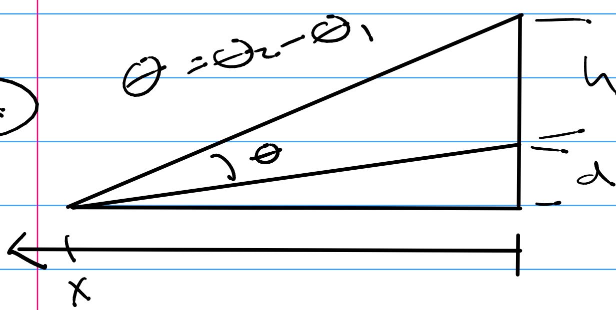
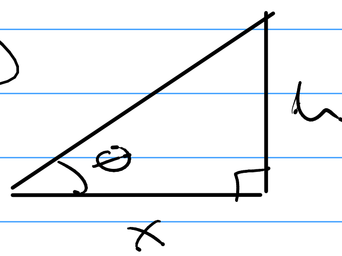
ex

$\theta = \theta_2 - \theta_1$



1. $\theta(x)$ — maximize this?
2. domain
3. $\boxed{\theta'(x)}$ why? critical numbers
4. deriv. tests

Compare to ...



$\tan\theta = \frac{h}{x}$ $\qquad$ $\cot\theta = \frac{x}{h}$

$\theta = \arctan\left(\frac{h}{x}\right)$

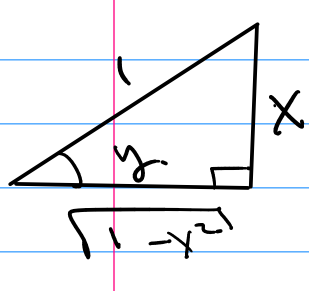
$\theta(x) = \arctan(h/x)$ $\qquad$ $\theta(x) = \text{arccot}\left(\frac{x}{h}\right)$

$0 < x < +\infty$

Last Time: $y = \sin(x)$ $\quad -\pi/2 \le x \le \pi/2$ $\quad -1 \le y \le 1$

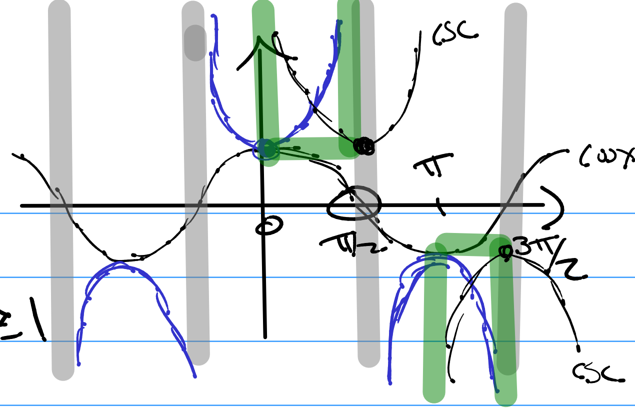
$x = \sin(y)$

$y = \sin^{-1}\left(\frac{x}{1}\right)$ $\quad -1 \le x \le 1, \quad -\pi/2 \le y \le \pi/2$



→ implicit deriv. $\quad D_x\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$

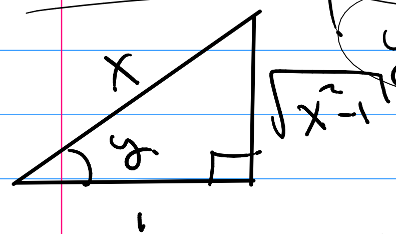
ex $y = \sec x = \dfrac{1}{\cos x}$



$y = \sec x \qquad 0 \le x < \pi/2, \quad |y| \ge 1$
$\qquad\qquad\quad \pi \le x < 3\pi/2$

Invert

$x = \sec(y) \quad |x| \ge 1, \quad 0 \le y < \pi/2$
$y = \sec^{-1}(x) \qquad\qquad\quad \pi \le y < 3\pi/2$

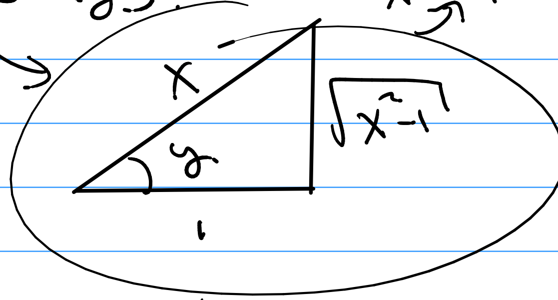


$D_x[\sec^{-1}(x)] = ?$

Implicit Form: $\sec(y) = x$

$$\sec(y)\tan(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec(y)\tan(y)} = \frac{1}{x\sqrt{x^2-1}}$$



So $D_x[\sec^{-1}(x)] = \dfrac{1}{x\sqrt{x^2-1}}$

① $y = \sin^{-1}(x)$

$$D_x[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

② $y = \cos^{-1}(x)$

$$D_x[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$$

③ $y = \tan^{-1}(x)$

$$D_x[\tan^{-1}(x)] = \frac{1}{1+x^2}$$

④ $y = \cot^{-1}(x)$

$$D_x[\cot^{-1}(x)] = \frac{-1}{1+x^2}$$

⑤ $y = \sec^{-1}(x)$

$$D_x[\sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}$$

⑥ $y = \csc^{-1}(x)$

$$D_x[\csc^{-1}(x)] = \frac{-1}{x\sqrt{x^2-1}}$$

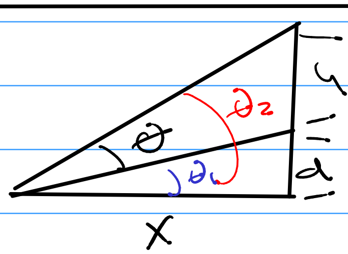
and

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$$

Back to...



$$\theta = \theta_2 - \theta_1$$

θ_2 : $\cot \theta_2 = \frac{x}{4+d} \rightarrow \theta_2(x) = \cot^{-1}\left(\frac{x}{4+d}\right)$

θ_1 : $\theta_1(x) = \cot^{-1}\left(\frac{x}{d}\right)$

$$\Theta(x) = \cot^{-1}\left(\frac{x}{h+d}\right) - \cot^{-1}\left(\frac{x}{d}\right) \quad 0 < x < +\infty$$

Maximize?

$$\Theta'(x) = \frac{-1}{1+\left(\frac{x}{h+d}\right)^2} \cdot \left(\frac{1}{h+d}\right) + \frac{1}{1+\left(\frac{x}{d}\right)^2} \cdot \left(\frac{1}{d}\right)$$

$$\Theta'(x) = \frac{-1}{(h+d) + \frac{x^2}{(h+d)}} + \frac{1}{d + \frac{x^2}{d}} \stackrel{?}{=} 0$$

Finish

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ex
$$\int \frac{1}{x\sqrt{x^2-4}}\,dx \qquad \text{know} \quad \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x)+C$$

$$x^2-4 = 4\left(\frac{x^2}{4}-1\right) = 4\left(\left(\frac{x}{2}\right)^2-1\right)$$

$$\frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx = \boxed{\frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du}$$

let $u = x/2$

$du = \frac{1}{2}dx$

$$= \frac{1}{2}\sec^{-1}(u)+C$$

$$= \boxed{\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+C}$$

$$\boxed{\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C}$$