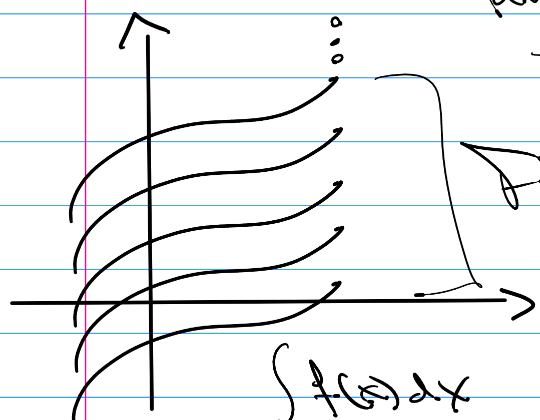
Math 243

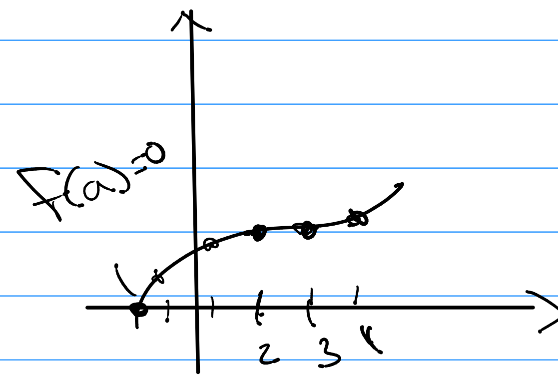
$\boxed{7.7}$ Using approx integrals to create Antiderivatives

$$\int_a^b f(x)\,dx \approx \boxed{\text{any approx tech.}}$$

use Fund. Th^{m}

$$\boxed{F(x) = \int_a^x f(t)\,dt \quad \text{where } F(a) = 0}$$

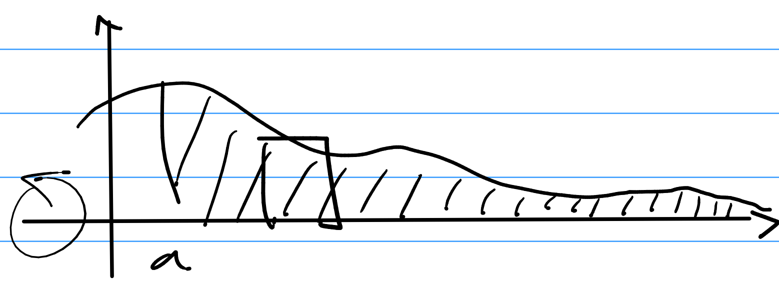




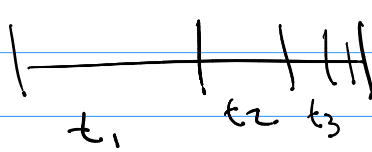
$\boxed{7.8}$ Improper Integrals $\int_a^b f(x)\,dx$ area

$a = -\infty$, $b = +\infty$, or both

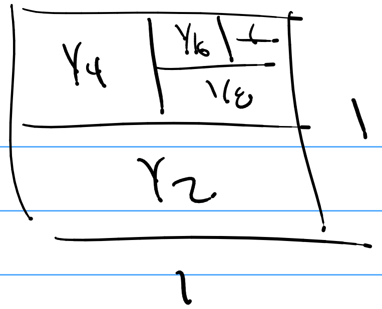
Idea:



$$\int_a^{+\infty} f(x)\,dx$$



$t_1 + t_2 + t_3 + t_{16} + \cdots$



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\boxed{\cdots}=1$$ conv.

$$\underbrace{\frac{1}{2}+\frac{1}{3}}+\underbrace{\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}=+\infty$$ divergent.

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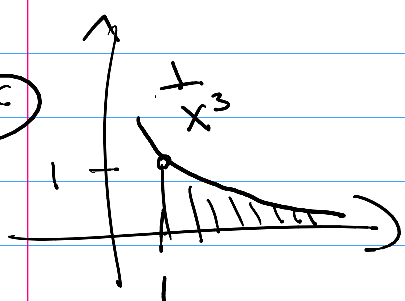
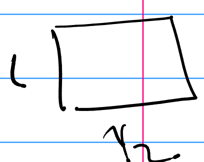
(1) $\displaystyle\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx$ $\quad f(x)$ is defined on a to $+\infty$

ex



$$\int_1^{\infty}\frac{1}{x^3}\,dx = \lim_{t\to+\infty}\left(\int_1^t \frac{1}{x^3}\,dx\right)$$



$$= \lim_{t\to+\infty}\left[-\frac{1}{2}x^{-2}\Big|_1^t\right] = \lim_{t\to+\infty}\left[-\frac{1}{2t^2}+\frac{1}{2}\right]$$

$$= 0+\frac{1}{2} = \boxed{\frac{1}{2}}$$

If limit $\boxed{\text{does not exist}}$ we say $\displaystyle\int_a^{+\infty} f(x)\,dx$ diverges.

does not exist

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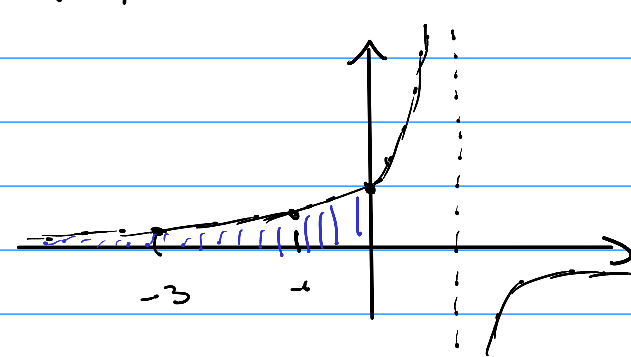
(2) $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{x\to-\infty}\int_t^b f(x)\,dx$

ex $\displaystyle\int_{-\infty}^{0}\frac{1}{1-x}\,dx$



$$\int_{-\infty}^{0} \frac{1}{1-x}\,dx = \lim_{t\to-\infty}\left[\int_{t}^{0} \frac{1}{1-x}\,dx\right] = \lim_{t\to-\infty}\left[-\ln|1-x|\Big|_{t}^{0}\right]$$

$$= \lim_{t\to-\infty}\left[0 + \ln|1-t|\right]$$

$$= \lim_{t\to-\infty} \ln|1-t| = \lim_{t\to+\infty} \ln(1+t) = +\infty$$

So $\int_{-\infty}^{0} \frac{1}{1-x}\,dx$ diverges.

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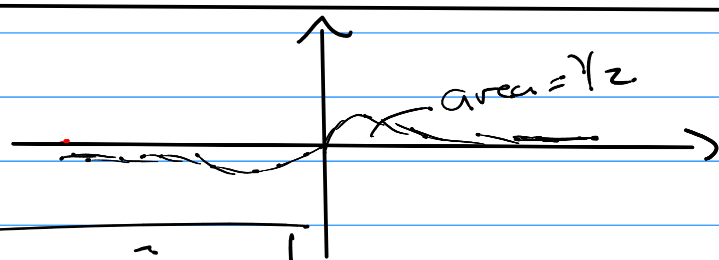
(3) $$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\boxed{\int_{-\infty}^{a} f(x)\,dx}}_{\text{type 2}} + \underbrace{\boxed{\int_{a}^{+\infty} f(x)\,dx}}_{\text{type 1}}$$

Note: $f(x)$ must be defined on $-\infty, +\infty$

Ex $$\int_{-\infty}^{\infty} x\cdot e^{-x^2}\,dx$$



$$\int_{-\infty}^{0} x e^{-x^2}\,dx + \boxed{\int_{0}^{+\infty} x e^{-x^2}\,dx}$$

$$\boxed{\int_{0}^{+\infty} x e^{-x^2}\,dx} = \lim_{t\to+\infty}\left[\int_{0}^{t} x e^{-x^2}\,dx\right]$$

$x=0 \to u=0$

$x=t \to u=-t^2$

$u=-x^2 \quad du=-2\boxed{x\,dx}$

$$= \lim_{t\to+\infty}\left[-\frac{1}{2}\int_0^{-t^2} e^u\,du\right] = \frac{1}{2}\lim_{t\to+\infty}\left[\int_{-t^2}^{0} e^u\,du\right]$$

$$= \frac{1}{2}\lim_{t\to+\infty}\left[e^u\Big|_{-t^2}^{0}\right] = \frac{1}{2}\lim_{t\to+\infty}\left[1-\cancel{e^{-t^2}}^{\,0}\right]$$

$$= \frac{1}{2}\left[1-0\right] = \frac{1}{2}$$

left side $\displaystyle\int_{-\infty}^{0} x e^{-x^2}\,dx \overset{\text{by sym.}}{=} -\frac{1}{2}$

$\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$ net signed area

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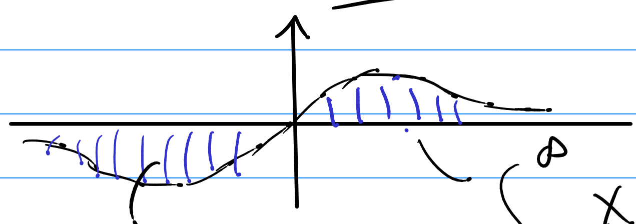
ex area **between** xe^{-x^2} and x-axis $= \boxed{1}$



$\displaystyle\int_0^{\infty} x e^{-x^2}\,dx = \frac{1}{2}$

$\displaystyle\int_{-\infty}^{0} 0 - x e^{-x^2}\,dx = \frac{1}{2}$

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type 1 improper

$\displaystyle\int_a^b f(x)\,dx$ $\quad a=-\infty$ or $b=+\infty$ or both type 2 improper

$\int_a^b f(x)\,dx$ $f(x)$ in interval $[a,b]$ has a vert. asymptote.