Math 293

Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \ \text{convergent}$$

Convergence: study $\sum_{n=1}^{K} a_n = a_1 + a_2 + \dots + a_K = S_K$

$\lim_{K \to \infty} S_K = S \rightarrow$ say $\sum a_n = S$ convergent series

ex) geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r| < 1 \quad \text{divergent otherwise}}$$

Useful parts of this...

① Sum $\quad \sum_{n=0}^{K} ar^n = a\left(\frac{1-r^{K+1}}{1-r}\right)$

② $\quad \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \ |x|<1$

Other Series

$\{a_n\} : \{a_1, a_2, a_3, a_4, \dots\}$

① Telescoping Series

ex) $\sum_{n=1}^{\infty} (a_{n+1}) - (a_n)$

$$= \underbrace{(a_2 - a_1)}_{n=1} + \underbrace{(a_3 - a_2)}_{n=2} + \underbrace{(a_4 - a_3)}_{n=3} + \dots$$

ex $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

Consider $$S_K=\sum_{n=1}^{K}\left(\frac{1}{n}-\frac{1}{n+1}\right)=$$

$$=\underbrace{\left(1-\frac{1}{2}\right)}_{n=1}+\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=2}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=3}+\dots+\underbrace{\left(\frac{1}{K-1}-\frac{1}{K}\right)}_{n=K-1}+\underbrace{\left(\frac{1}{K}-\frac{1}{K+1}\right)}_{n=K}$$

$$=1-\frac{1}{K+1}$$

So $S_K=1-\frac{1}{K+1}$

$$\lim_{K\to\infty}1-\frac{1}{K+1}=1 \quad \text{so} \quad \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$$

Note! $$\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)} \qquad \sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

or $$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\dots=1$$

$$\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}+\frac{2}{35}+\dots$$

$$\frac{2}{n^2-1}=\frac{2}{(n+1)(n-1)}=\frac{A^{=-1}}{n+1}+\frac{B^{=1}}{n-1}=\boxed{\frac{1}{n-1}-\frac{1}{n+1}}$$

$$2=A(n-1)+B(n+1)$$

$$\sum_{n=2}^{\infty}\frac{1}{n-1}-\frac{1}{n+1}=?$$

$$S_K=\sum_{n=2}^{K}\frac{1}{n-1}-\frac{1}{n+1}=\underbrace{\left(1-\frac{1}{3}\right)}_{n=2}+\underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{n=3}+\underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{n=4}+\dots$$

$$S_k = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots$$

$$\cdots + \left(\frac{1}{k-4} - \frac{1}{k-2}\right) + \left(\frac{1}{k-3} - \frac{1}{k-1}\right) + \left(\frac{1}{k-2} - \frac{1}{k}\right) + \left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

$n = k$

$$S_k = 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1}$$

$$\lim_{k\to\infty} S_k = 1 + \frac{1}{2} = \frac{3}{2}$$

$\boxed{\text{th}^{m}}$ if $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

logical equivalence

if a, then b is logically the same as if not b, then not a

contrapositive

Contrapositive:

$\boxed{\text{if } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum a_n \text{ is divergent}}$

Divergence Test.

ex $$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{9n^2+2n-3}} = \frac{3}{\sqrt{8}} + \frac{5}{\sqrt{37}} + \cdots$$

$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{9n^2+2n-3}} = \lim_{n\to\infty} \frac{(2n+1)\frac{1}{n}}{\sqrt{9n^2+2n-3}\left(\sqrt{\frac{1}{n^2}}\right)}$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{9+\frac{2}{n}-\frac{3}{n^2}}} = \frac{2}{\sqrt{9}} = \frac{2}{3} \neq 0$$

so series div.

ex $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

$\lim_{n\to\infty} \frac{1}{n} = 0$ div. test doesn't apply.

Next:

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$S_4 = 1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}$

$\vdots$

$S_8 = 1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$

$\vdots$

$S_{16} = 1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}$

$\underbrace{\frac{1}{4} + \frac{1}{4}}_{= \frac{1}{2}} \quad \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{= 1/2} \quad \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_{= 1/2}$

$S_8 = S_{2^3} > 1 + 3\left(\frac{1}{2}\right)$

$S_{16} = S_{2^4} > 1 + 4\left(\frac{1}{2}\right)$

$S_{32} = S_{2^5} > 1 + 5\left(\frac{1}{2}\right)$

$S_{2^k} > \underline{1 + k\left(\frac{1}{2}\right)} \to \infty$ as $k \to \infty$

harmonic sum

so $\boxed{1 + \frac{1}{2} + \frac{1}{3} + \ldots \text{ is div.}}$

If $\sum a_n$, $\sum b_n$ converge

then $\sum a_n \pm b_n = \sum a_n \pm \sum b_n$

$$\sum c a_n = c \sum a_n$$
