Math 243

Seq: conv. if $\lim_{n\to\infty} a_n = L$

Series: ① Do the partial sum work..

$\sum a_n = ?$

a) $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k$

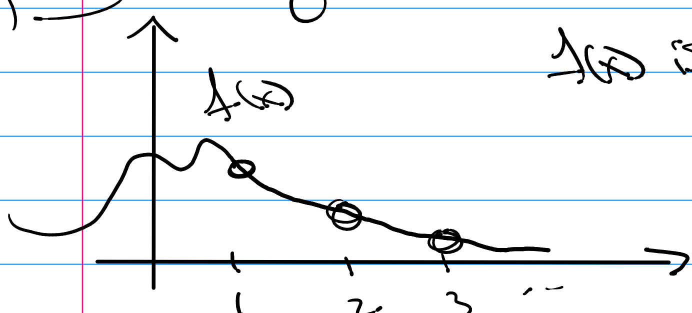
b) $\lim_{k\to\infty} S_k = S = \sum a_n$ conv.

div → $\sum a_n$ div.

② Divergence test.

if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

11.1 Integral test for $\sum a_n$

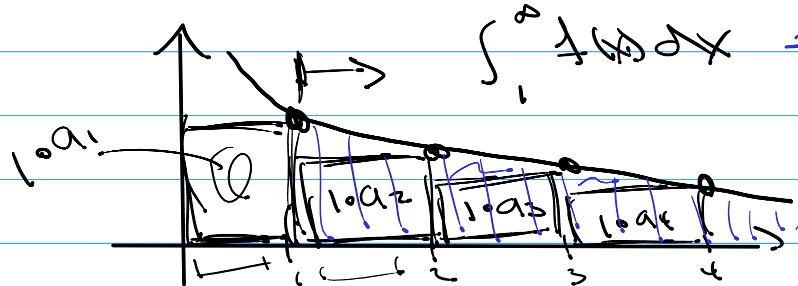


f(x) is ① continuous ② positive ③ decreasing for $x \geq 1$

and $f(n) = a_n$ for $n = 1, 2, 3, \dots$

now consider $a_1 + a_2 + a_3 + a_4 + \dots$

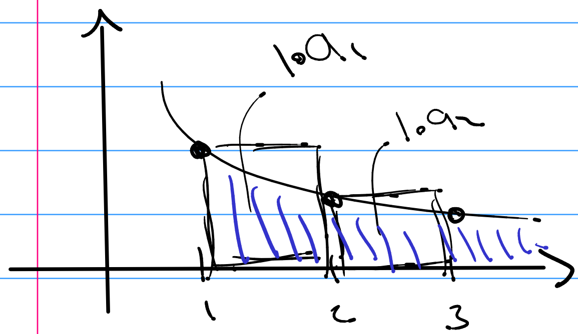
$= 1\cdot a_1 + 1\cdot a_2 + 1\cdot a_3 + 1\cdot a_4 + \dots$



$\int_1^\infty f(x)\,dx$ = area under f(x) from $x=1$ to ∞

Improper integral.

So if $\int_1^\infty f(x)$ conv. then $\sum a_n$ conv.



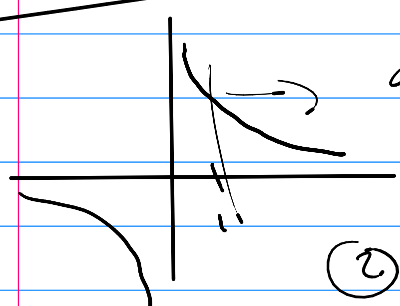
$\int_1^\infty f(x)\,dx$ diverges then $\sum a_n$ diverges.

So $\int_1^\infty f(x)\,dx$ converges iff $\sum a_n$ converges

$\int_1^\infty f(x)\,dx$ diverges iff $\sum a_n$ diverges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \cdots$$ **div**

Integral Test $\quad f(x) = \frac{1}{x}$



on $x \geq 1$ need ① cont. $\frac{1}{x}$ is only disc. @ $x=0$ **OK**

② positive $\frac{1}{x} > 0$ **OK**

- 0 +

③ Dec. $f'(x) = \frac{-1}{x^2} < 0$ always OK

So check $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t = \lim_{t\to\infty} \ln(t) = \infty$$

$\underline{\text{div}}$

$$\boxed{\text{so } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is } \boxed{\text{also}} \text{ div.}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p > 0 \qquad \text{on } x \geq 1$$

let $f(x) = \frac{1}{x^p}$ ① $\underline{\text{Cont?}}$ $\boxed{\text{ok}}$

② $\underline{\text{pos?}}$ $\boxed{\text{ok}}$

③ $\underline{\text{dec?}}$ $f'(x) = -p\, x^{-p-1} = \frac{-p}{x^{p+1}}$ $\boxed{\text{ok}}$

$f' < 0$

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t=\infty} \boxed{\int_1^t x^{-p}\, dx}$$

$$\text{So } \int \frac{1}{x^p}\, dx = \begin{cases} \ln|x| & p = 1 \\ \frac{1}{1-p} x^{1-p} & p \neq 1 \end{cases}$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^p}\, dx = \begin{cases} \lim_{t\to\infty} \ln|x| \Big|_1^t & p = 1 \\ \lim_{t\to\infty} \frac{1}{1-p} x^{1-p} \Big|_1^t & p \neq 1 \end{cases}$$

$\underline{\text{div.}}$ (above)

$$\lim_{t\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^t = \frac{1}{1-p} \lim_{t\to\infty} \left[t^{1-p} - 1 \right]$$

heart of the problem $\lim_{t\to\infty} t^{1-p}$ for p's

a) if $\boxed{p > 1}$ $1-p$ <u>negative.</u> $t^{(1-p)\text{ neg.}} = \left(\frac{1}{t}\right)^{(p-1)\text{ pos}}$

and $\lim_{t\to\infty} \frac{1}{t} = 0$ $\boxed{\text{conv}}$

b) if $\boxed{p < 1}$ $(1-p)$ pos $t^{(1-p)\text{ pos}}$

and $\lim_{t\to\infty} t^{\text{pos}} = \infty$ $\boxed{\text{div}}$

so $\int_1^\infty \frac{1}{x^p}\,dx$ is conv. if $p > 1$

div if $p \le 1$

$$\therefore \boxed{\sum \frac{1}{n^p} \text{ is conv. if } p > 1 \quad \text{div if } p \le 1}$$ p-series test

ex $\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$ is conv. b/c $7/6 > 1$

ex $\sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ is div b/c $5/6 < 1$

Consider if $\sum_{n=1}^{\infty} a_n = S$ (conv.)

So $S = a_1 + a_2 + a_3 + a_4 + \dots + S_k + S_{k+1} + S_{k+2} \dots$

approx $S \approx a_1 + a_2 + a_3 + \dots + a_k = S_k$

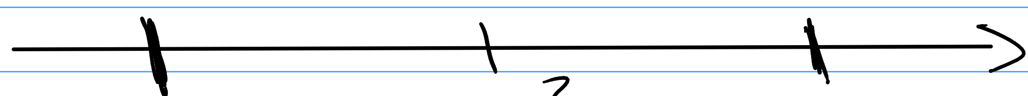
$S = S_k + $ error

remainder

Define: $S - S_k = S_{k+1} + S_{k+2} + \dots = R_k$

$$\int_{k+1}^{\infty} f(x)\,dx \le R_k \le \int_{k}^{\infty} f(x)\,dx$$

or $$S_k + \int_{k+1}^{\infty} f(x)\,dx \le S \le S_k + \int_{k}^{\infty} f(x)\,dx$$



$S_k = a_1 + a_2 + \dots + a_k$ $+ \int_{k+1}^{\infty} f(x)\,dx$

$S = ?$

$S_k + \int_{k}^{\infty} f(x)\,dx$