Math 243

Q's $\frac{1}{1\sqrt{1}} + \frac{2}{2\sqrt{2}} + \frac{3}{3\sqrt{3}} + \frac{4}{4\sqrt{4}} + \dots + a_n + \dots$

$$a_n = \frac{n}{n\sqrt{n}} = \frac{n^1}{n^1 n^{1/2}} = \frac{n^1}{n^{3/2}} = \frac{1}{n^{1/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series $p = \frac{1}{2} \to$ div ?

$p \le 1 \to$ div.
$p > 1 \to$ conv.

(a) 11.2 #47 $\sum_{n=1}^{\infty} e^{1/n} - e^{1/n+1}$ (book)

skip

11.1

#47 $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$

let $f(x) = \left(1+\frac{3}{x}\right)^x$ (b/c $f(n) = a_n$)

So $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$ type 1^∞

use ① $\ln(a^b) = b\ln a$
② $\exp(\ln(\square)) = \square$

$$\text{so } \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} \exp\left(\ln\left(\left(1+\frac{3}{x}\right)^x\right)\right)$$

$$= \lim_{x\to\infty} \exp\left(x \ln\left(1+\frac{3}{x}\right)\right)$$

$$= \exp\left(\lim_{x\to\infty} \overset{\to\infty}{x} \, \overset{\to 0}{\ln\left(1+\frac{3}{x}\right)}\right)$$ type $\infty \cdot 0$

$$= \exp\left(\lim_{x\to\infty} \frac{\ln(1+\frac{3}{x})}{\frac{1}{x}} \right)$$ type $\frac{0}{0}$ use L'Hospitals

$$= \exp\left(\lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot \left(\frac{-3}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} \right)$$

$$= \exp\left(\lim_{x\to\infty} 3 \frac{1}{1+\cancel{\frac{3}{x}}} \right) = \exp(3) = e^3$$

11.4 #27

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-2n}$$

conv. $|r|<1$

geometric $a\cdot(r)^n$

Comparison

$$\frac{(1+\frac{1}{n})^2}{e^{2n}} \sim \frac{1}{e^{2n}} = 1\left(\frac{1}{e^2}\right)^n$$ $a=1$, $r=\frac{1}{e^2}$

for large n conv.

$\boxed{\text{Guess conv.}}$ and act like $\left(\frac{1}{e^2}\right)^n = \frac{1}{e^{2n}}$

Comparison test vs limit comparison test

$$\frac{(1+\frac{1}{n})^2}{e^{2n}} < (?)$$ conv!

$$\lim_{n\to\infty} \frac{\frac{(1+\frac{1}{n})^2}{e^{2n}}}{\frac{1}{e^{2n}}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1$$

pos. constant

these act alike

$\therefore$ b/c $\sum \frac{1}{e^{2n}}$ conv. $\to \sum \frac{(1+\frac{1}{n})^2}{e^{2n}}$ conv error estimates:

Integral test $\quad S = a_1 + a_2 + \dots + a_k + R_k$

$$S = S_k + R_k$$

error $\quad |S - S_k| = |R_k|$ size of remainder

by integral test $\quad R_k \le \int_k^\infty f\,dx$ $\quad f(n) = a_n$

Comparison test. $\quad \sum a_n \,,\ \sum b_n$ conv.
$a_n \le b_n$ $\quad$ known series

showed $\sum a_n$ conv. by comparing to $\sum b_n$

Say:
$$S = \underbrace{a_1 + a_2 + \dots + a_k}_{S_k} + R_k$$
$$t = \underbrace{b_1 + b_2 + \dots + b_k}_{t_k} + T_k$$

$\rightarrow\ R_k = S - S_k \qquad T_k = t - t_k$

but $\quad R_k = a_{k+1} + a_{k+2} + \dots$ $\quad$ knew $a_n \le b_n$
$\quad T_k = b_{k+1} + b_{k+2} + \dots$

$\rightarrow\ R_k \le T_k \le \int_k^\infty f\,dx$ $\quad f(n) = b_n$

from known $\sum b_n$

ex $\sum \frac{(1+\frac{1}{n})^2}{e^{2n}}$ is conv. b/c $\sum \frac{1}{e^{2n}}$ is conv.

10-term estimate $S \approx \frac{(1+\frac{1}{1})^2}{e^2} + \frac{(1+\frac{1}{2})^2}{e^4} + \dots + \frac{(1+\frac{1}{10})^2}{e^{20}}$

$$R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{1}{e^{2x}}\,dx$$

uses $\frac{(1+\frac{1}{n})^2}{e^{2n}}$ uses $\frac{1}{e^{2n}} \to f(x) = \frac{1}{e^{2x}}$

$$\int_{10}^{\infty} e^{-2x}\,dx = \lim_{t\to\infty} \int_{10}^{t} e^{-2x}\,dx = \lim_{t\to\infty} \int_{-20}^{-2t} \frac{1}{-2} e^{u}\,du$$

$u = -2x$

$du = -2dx$

$$= \frac{1}{2} \lim_{t\to\infty} \int_{-2t}^{-20} e^{u}\,du = \frac{1}{2} \lim_{t\to\infty} e^{u} \Big|_{-2t}^{-20}$$

$$= \frac{1}{2} \lim_{t\to\infty} \left[e^{-20} - e^{-2t} \right] = \frac{1}{2} \lim_{t\to\infty} \left[\frac{1}{e^{20}} - \cancel{\frac{1}{e^{2t}}} \right]$$

0

$$= \frac{1}{2e^{20}} \approx 0.000000001$$

11.5 $\sum a_n$ ← pos and/or negative?

ex **Alternating Series**

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

or

$$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \dots$$

Use $(-1)^n$, $n = 0, 1, 2, \ldots$

Seq: $1, -1, 1, -1, 1, -1, \ldots$

or $(-1)^{n+1}$, $n = 1, 2, 3, \ldots$

Seq: $1, -1, 1, -1, 1, -1, \ldots$

or $(-1)^{n-1}$, $n = 1, 2, 3, \ldots$

Seq $1, -1, 1, -1, 1, -1, \ldots$

or $(-1)^n$ $n = 1, 2, 3, \ldots$

Seq: $-1, 1, -1, 1, -1, 1, \ldots$

Ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = (-1)^2 \frac{1}{1} + (-1)^3 \frac{1}{2} + (-1)^4 \frac{1}{3} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = (-1)^0 \frac{1}{0+1} + (-1)^1 \frac{1}{1+1} + (-1)^2 \frac{1}{2+1} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

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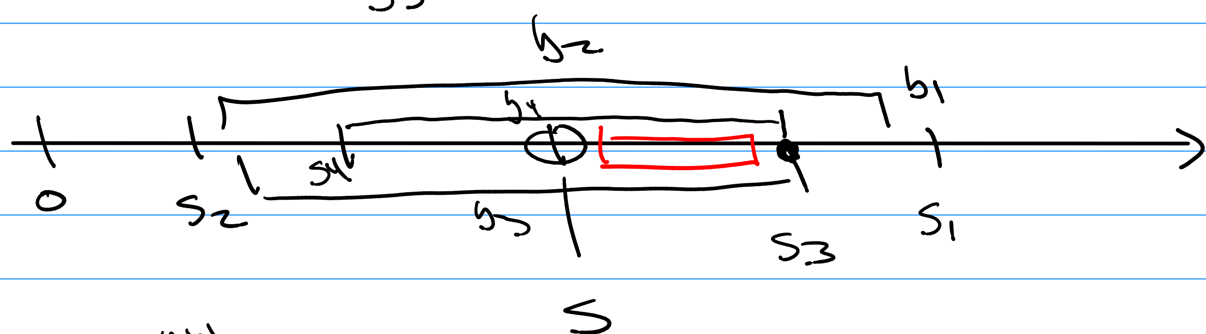
alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots$

(F) ① b_n are dec. $b_n \geq b_{n+1}$

② $\lim\limits_{n\to\infty} b_n = 0$

$$S = \underbrace{\underbrace{\underbrace{b_1}_{S_1} - b_2}_{S_2} + b_3}_{S_3} - b_4 + b_5 - b_6 + \dots$$



$\sum (-1)^{n+1} b_n$ is convergent

$S_3 \approx S$ see $R_3 < b_4$

So $S \approx S_k$ and $R_k < b_{k+1}$