Math 243

Alt. Series: $b_1 - b_2 + b_3 - b_4 + \dots$

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

Alt. Series test

$$\boxed{\begin{array}{l} \text{① } b_n \geq b_{n+1} \text{ (dec seq)} \\ \text{② } \lim_{n\to\infty} b_n = 0 \\ \text{then } \sum (-1)^{n+1} b_n \text{ converges.} \end{array}}$$

and $S_k \approx S$ error $R_k \leq b_{k+1}$

(Ex) **alt. harmonic** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

Alt. Series test ① $b_n = \frac{1}{n}$ $b_{n+1} = \frac{1}{n+1} \rightarrow \frac{1}{n} > \frac{1}{n+1}$ dec.

② $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ Converges

Q: Converges to ...? (approximate it to 4 decimals)

error < 0.0001

$$\text{error} = R_k \leq b_{k+1} = \frac{1}{k+1} < 0.0001$$

So $\frac{1}{k+1} < 10^{-5} \rightarrow k+1 > 10^5 \rightarrow k > 10^5 - 1$

if $K > 10^5 - 1$ let $K = 100{,}000$

$$S_{100,000} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots - \frac{1}{100,000}$$

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a_n sometimes pos/neg ex $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} = \frac{\sin(1)}{1^2} + \frac{\sin(2)}{2^2} + \frac{\sin(3)}{3^2} + \dots$

non-alternating addition and subtractions.

11.6 Absolute Convergence

consider $\boxed{\sum |a_n|}$ over $\sum a_n$

Def if $\sum |a_n|$ converges we will call $\sum a_n$ absolutely convergent

Def if $\sum |a_n|$ and it diverges

but $\sum a_n$ and it converges

we will call $\sum a_n$ conditionally convergent

Thm if $\sum |a_n|$ conv

then $\sum a_n$ conv.

ex $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} = \frac{\sin(1)}{1^2} + \frac{\sin(2)}{2^2} + \dots$

→ Consider $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right| = \frac{|\sin(1)|}{1^2} + \frac{|\sin(2)|}{2^2} + \dots$

$$a_n = \frac{|\sin(n)|}{n^2}$$

div test: $\lim_{n\to\infty} \frac{|\sin(n)|}{n^2} = 0$ ← test failed

by squeeze thm

$$\cancel{0} \le \frac{|\sin(n)|}{n^2} \le \cancel{\frac{1}{n^2}}$$

Note: $\frac{|\sin(n)|}{n^2} \le \frac{1}{n^2}$ ← conv. p-series $(p = 2 > 1)$

by comparison test b/c $\frac{1}{n^2}$ conv. $\boxed{\frac{|\sin(n)|}{n^2} \text{ conv.}}$

So we found $\sum \left| \frac{\sin(n)}{n^2} \right|$ conv.

call $\sum \frac{\sin(n)}{n^2}$ absolutely convergent

and by thm it converges

$\boxed{\text{ratio test}}$ $\sum a_n$ consider $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

① if $L < 1$ then $\sum a_n$ is absolutely conv.

→ $\sum a_n$ is conv.

② if $L > 1$ (includes ∞) then $\sum a_n$ is divergent

③ if $L = 1$ test fails → try another test ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ $\quad a_n = \frac{(-2)^n}{n^2} \quad a_{n+1} = \frac{(-2)^{n+1}}{(n+1)^2}$

ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}}$

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = \lim_{n\to\infty} \boxed{\frac{2^{n+1}}{2^n} \cdot \frac{n^2}{(n+1)^2}}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} 2 \cdot \left(\frac{n^2}{n^2+2n+1} \right)\frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} 2 \cdot \left(\frac{1}{(1 + \overset{0}{2/n} + \overset{0}{1/n^2})} \right) = 2 > 1$$

$\to \boxed{\text{div}}$

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ex $\displaystyle\sum_{n=0}^{\infty} \frac{100^n}{n!} = \frac{1}{1} + \frac{100}{1} + \frac{10{,}000}{2} + \frac{1{,}000{,}000}{6} + \ldots$

ratio test $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}} = \lim_{n\to\infty} \frac{100^{n+1}}{100^n} \cdot \frac{n!}{(n+1)!}$

$$= \lim_{n\to\infty} 100 \cdot \frac{\cancel{n(n-1)\cdots(1)}}{(n+1)\cancel{n(n-1)\cdots(1)}} = \lim_{n\to\infty} 100 \cdot \frac{1}{n+1}^{\,0}$$

$$= 0 < 1$$

$\boxed{\text{conv}}$