Math 243

11.6 Absolute Conv. (Ratio Test / Root Test

① $\sum |a_n|$ converges we say $\sum a_n$ is absolutely conv.
(also it is conv. as well)

② $\sum |a_n|$ diverges and $\sum a_n$ converges
we say $\sum a_n$ is conditionally convergent.

③ Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

a) $L<1$, then $\sum a_n$ is absolutely conv (and also conv.)

b) $L>1$, then $\sum a_n$ is divergent.

c) $L=1$, test fails → try something else.

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ex $\sum \frac{1}{n^3}$ (Know convergent p-series $3>1$)

ratio test:

$a_n = \frac{1}{n^3}$

$a_{n+1} = \frac{1}{(n+1)^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}}\right| = \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\cdot\frac{1/n^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^3}$$

$= 1$ if $L=1$ test fails
→ try something else!

ex $\sum_{n=0}^{\infty} \frac{100^n}{n!} = 1 + \frac{100}{1} + \frac{10{,}000}{2} + \dots$

ratio test: $\lim_{n\to\infty} \left| \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}} \right| = (\dots) = 0 < 1$ conv.

$\uparrow$ last class

ex $\sum_{n=1}^{\infty} \frac{n!}{n^n} = \frac{1}{1} + \frac{2\cdot 1}{2\cdot 2} + \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3} + \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4} + \dots$

try comparison test and I guess conv

Need: $\boxed{\frac{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}{n\cdot n\cdot n\cdots n\cdot n}} < \boxed{?}$ $\Leftarrow$ find something that is conv.

$$\frac{\cancel{n}\cdot(n-1)\cdot(n-2)\cdots(2)(1)}{\cancel{n}\; n \quad n \quad n\; n} < \frac{\cancel{n}\,(n-2)\cdots(2)(1)}{\cancel{n}\; n \cdots n\; n}$$

$$\dots < \frac{(2)(1)}{n\cdot n} = \frac{2}{n^2} \quad \text{conv.}$$

$\boxed{\text{ratio test}}$ $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \frac{(n+1)!}{n!}$

$a_n = \frac{n!}{n^n}$

$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$ $= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+\cancel{1}}} \frac{\cancel{(n+1)}\cancel{n!}}{\cancel{n!}}$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} \quad \text{know: } \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$L = \frac{1}{e} < 1$ so $\boxed{\text{abs conv } \underline{\text{and}} \text{ conv.}}$

$\boxed{\text{Root Test}}$ consider $\lim_{n\to\infty} (|a_n|)^{1/n} = L$

a) $L < 1$, then $\sum a_n$ is abs. conv. (and also conv)

b) $L > 1$, then $\sum a_n$ is div.

c) $L = 1$, test fails $\to$ try something else.

ex $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^n}$ $\quad a_n = \frac{(-3)^n}{n^n} \to |a_n| = \frac{3^n}{n^n}$

root test $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{3^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1$

so $\sum \frac{(-3)^n}{n^n}$ abs. conv (and conv.)

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ex $\sum_{n=1}^{\infty} \frac{1}{n^3}$

? $\boxed{\text{root test}}$ $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^3}\right)^{1/n} = \boxed{\lim_{n\to\infty} \frac{1}{n^{3/n}}} = L$

?

try $f(x) = \frac{1}{x^{3/x}} = x^{-3/x}$

$\lim_{x\to\infty} x^{-3/x}$ $\boxed{\text{type } \infty^0}$