Math 293

Power Series

C_i are real constants..

$$\boxed{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots}$$

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \dots \quad \text{power series}$$

have two big features
① converge
② diverge

we already have one of these

from $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$ geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{when } |r| < 1$$

let $r = x$ $a = 1$

$$\hookrightarrow \boxed{\underbrace{1 + x + x^2 + x^3 + \dots}_{\text{this is a power series}} = \frac{1}{1-x} \quad \text{when } |x| < 1}$$

we see $f_1(x) = 1 + x + x^2 + x^3 + \ldots$

$f_2(x) = \frac{1}{1-x}$

but the above says when $|x|<1$ $f_1 = f_2$

1st Power series centered at $x=a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

now let's study when some power series will conv / div

ex $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} x^n = -x + \frac{1}{\sqrt[3]{2}} x^2 - \frac{1}{\sqrt[3]{3}} x^3 + \frac{1}{\sqrt[3]{4}} x^4 - \ldots$$

ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ with $a_n = \frac{(-1)^n}{n^{1/3}} x^n$

$a_{n+1} = \frac{(-1)^{n+1}}{(n+1)^{1/3}} x^{n+1}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^{1/3}}}{\frac{x^n}{n^{1/3}}} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \left(\frac{n^{1/3}}{(n+1)^{1/3}} \right)$$

$$= \lim_{n\to\infty} |x| \cdot 1 = |x| = L$$

So we found $L = |x|$ know: $L<1 \to$ conv.

$L>1 \to$ div

$L=1 \to$??, test fails

if $|x| < 1$ our series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} x^n$ conv

if $|x| > 1$ our series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} x^n$ div

?? $|x| = 1$?? our series ...?? does what??

$\boxed{\text{Case 1}}$ $x = 1$ or $\boxed{\text{Case 2}}$ $x = -1$

Case 1

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} \cdot (1)^n$, where $(1)^n = 1$

$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/3}}$

this a conv. alt. series!

→ conv. @ $x = 1$

Case 2

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} \cdot (-1)^n$

$= \sum_{n=1}^{\infty} \underbrace{(-1)^{2n}}_{=1} \frac{1}{n^{1/3}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

divergent p-series

div @ $x = -1$

$\boxed{\text{So}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} x^n = -x + \frac{1}{\sqrt[3]{2}} x^2 - \frac{1}{\sqrt[3]{3}} x^3 + \frac{1}{\sqrt[3]{4}} x^4 - \dots$

conv: $-1 < x \le 1$

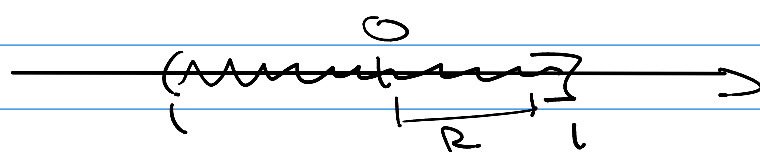
div: $x \le -1$ or $x > 1$

Terms

(1) the series was about $x = 0$

(2) radius of conv. was $R = 1$

(3) interval of conv.



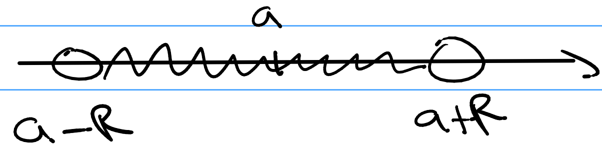
Thm $\sum c_n(x-a)^n$ power series about $x=a$ will have only one of the three possibilites

① conv only at $x=a$

② conv for all reals $x \in \mathbb{R}$ from $-\infty$ to ∞ $R=\infty$

③ a) Radius of conv. of R = finite number

b) interval of conv



c) and you need to test $x=a-R$
$x=a+R$
individually to know if they conv/div there.

Ex

$$\sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^{2n} = \underset{n=0}{1} + \underset{n=1}{\frac{1}{1!}(x-3)^2} + \underset{n=2}{\frac{1}{2!}(x-3)^4} + \underset{n=3}{\frac{1}{3!}(x-3)^6} + \dots$$

Conv?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ $(f(x))$

So $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}(x-3)^{2n+2}}{\frac{1}{n!}(x-3)^{2n}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{|x-3|^{2n+2}}{|x-3|^{2n}}$

$= \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)(\cancel{n!})} \cdot |x-3|^2 = \lim_{n\to\infty} \frac{|x-3|^2}{n+1} = |x-3|^2 \cdot \lim_{n\to\infty} \frac{1}{n+1}$

$= |x-3|^2 \cdot 0 = 0 = L$

So our $L = 0 < 1$ for all x.

So abs conv. for all x

$\therefore \sum_{n=1}^{\infty} \frac{1}{n!}(x-3)^n$ $R = \infty$ interval is $(-\infty, \infty)$