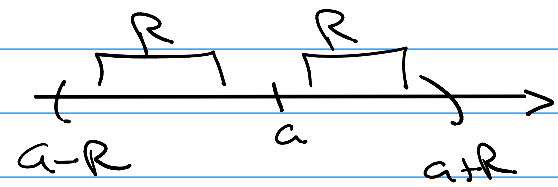
Math 243

Power Series as a function

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

abs. conv on



R : radius of convergence

$(a-R, a+R)$ interval of conv with $a-R$ and $a+R$ are special cases to check for convergence

Function : Domain $\to$ Codomain

$f(x) = y$ or $x \xrightarrow{f} y$ or (x,y) is in the function

So we can see that $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a function on its natural domain.

ex $\sum_{n=0}^{\infty} \frac{1}{n!}(x-2)^n = 1 + (x-2) + \frac{1}{2!}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \cdots$

Interval of conv $\equiv$ domain of the function

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)!}}{\frac{(x-2)^n}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot |x-2|$$

$= 0 \cdot |x-2| = 0 = L$ obviously < 1

So $R=\infty$ So domain is all $\mathbb{R}$.

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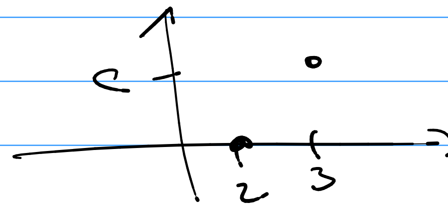
ex $x=3 \to y=? = \sum_{n=0}^{\infty} \frac{1}{n!}(3-2)^n = 1+1+\frac{1}{2}+\frac{1}{3!}+\dots$

$x=3 \to y=e$



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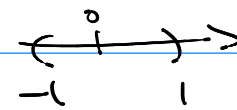
we can say $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ with domain $(a-R, a+R)$

Also we already had.

$$f(x)=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots \quad \text{domain } |x|<1$$



is also

$$f(x)=\frac{1}{1-x} \quad \text{only on } |x|<1$$

Note: on knowing domains

$$\boxed{\frac{x+3}{x^2-9}} = \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x-3)} = \boxed{\frac{1}{x-3},\ x\neq -3}$$

domain: $x\neq 3$, $x\neq -3$

domain $x\neq 3$

Properties of equality

① $a=b$, $b=a$

② we can substitute (replace with equal things)

ex $a=b+2$

given $a^2+\sqrt{a}$

$=(b+2)^2+\sqrt{b+2}$

③ $a=b$

a) $\frac{d}{dx}[a]=\frac{d}{dx}[b]$

b) $\int\{a\}dx = \int\{b\}dx$

Make some new $f_1(x)=\boxed{f_2(x)}$ ← power series

Start $\boxed{\frac{1}{1-x}=1+x+x^2+x^3+\ldots \quad \text{only on } |x|<1}$

Use it $1+(-\tfrac{1}{2})+(-\tfrac{1}{2})^2+(-\tfrac{1}{2})^3+\ldots$

$=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$

$=\frac{1}{1-(-\frac{1}{2})}=\frac{1}{3/2}=\boxed{\frac{2}{3}}$

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots \qquad |\square| < 1$$

let $\square = 1 - x$

$$\Rightarrow \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \dots$$

only if $|1-x| < 1$

$$\Rightarrow \frac{1}{x} = 1 + (1-x) + (1-x)^2 + \dots \quad \text{on } |1-x|<1$$

$$-1 < 1-x < 1$$

$$-2 < -x < 0$$

$$2 > x > 0$$

on $x \in (0,2)$ $\quad \frac{1}{x} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \dots$

ex $\quad \frac{1}{\sqrt{2}} = 1 + (1-\sqrt{2}) + (1-\sqrt{2})^2 + \dots$

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \dots \qquad |\square| < 1$$

sub $\quad$ let $\square = (x^2)$

$$\Rightarrow \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots \qquad |x^2| < 1$$

domain $\boxed{|x| < 1}$

Sub $\square = -x^2$

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots \qquad |\square| < 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \qquad |x| < 1$$

Th^m $f(x) = \underline{\sum_{n=0}^{\infty} c_n (x-a)^n}$ with radius of conv. of R

then: $\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[c_0 + c_1(x-a) + c_2(x-a)^2 + \dots\right]$

$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$ is also conv. with R

and $\int f(x)dx = \int \left[c_0 + \underline{c_1(x-a)} + \dots\right]dx$

$$= c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \dots$$

+ Constant of integration

$$= \left[\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right] + C$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{(n+1)}$$

conv on R, Radius on conv.

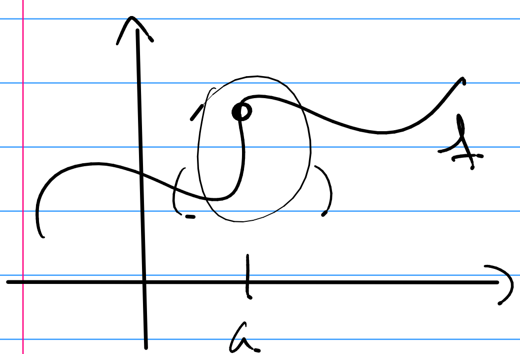
fund

Ex $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^7 - \dots$ on $|x|<1$

Know: #1 $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$

#2 $\displaystyle\int (1 - x^2 + x^4 - x^6 + x^8 - \dots)\,dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$

so $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots \quad |x|<1$



$\dfrac{1}{1-\square} = 1 + \square + \square^2 + \dots$