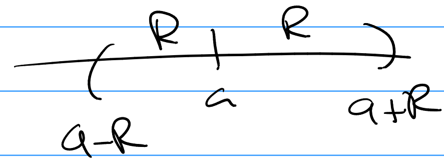
Math 243

power series as $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \ldots$

domain for



and f $f(x)$ = some other expression of a function ..

till now ... play with $\frac{1}{1-\square} = 1 + \square + \square^2 + \ldots \quad |\square| < 1$

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another: $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

$R = ?$ use ratio test $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot |x|$

$= 0 \cdot |x| = 0 < 1$ always $R = \infty$

So $f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$ domain: all $\mathbb{R}$

notice $f' = 0 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

$\boxed{f' = f}$ know from exponential chapter

$f(x) = Ke^x$

but our function is $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$ @ 0 $\Rightarrow f(0) = 1$

and $Ke^0 = K$ so $K = 1$

$\boxed{\text{New equality}}$ $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$ domain all $\mathbb{R}$.

use of this...

before $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ $e \approx \left(1+\frac{1}{n}\right)^n$ for large n

now $e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \frac{1}{5!} + \frac{1}{6!} + \dots$

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$e \approx \frac{48 + 12 + 4 + 1}{24} = \frac{65}{24}$$

or $e^3 = 1 + 3 + \frac{1}{2!}(3)^2 + \frac{1}{3!}(3)^3 + \dots$

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what about $e^{100} = ?$

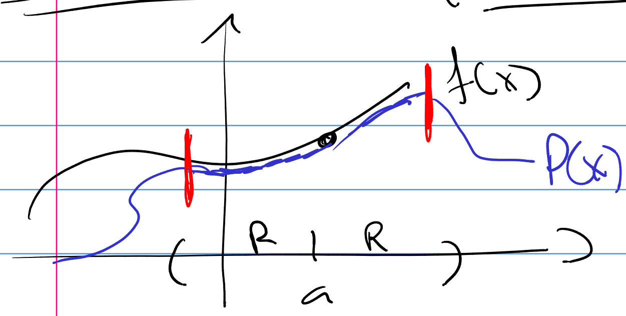
use $\left(e^{x/2}\right)^2 = e^x$

$$e^{100} - \left(e^{50}\right)^2 = \left(\left(e^{25}\right)^2\right)^2 = ? \text{ etc?}$$

So we at least have some functions $\frac{1}{1-x}$, e^x, others based on these

where they equal $\sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of conv, R

Next Question... $\boxed{11.10}$



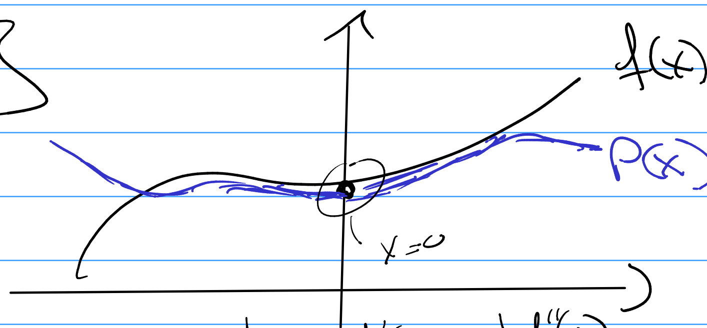
given $f(x)$

want: $f(x) = \big($ power series about $x=a \big)$ with radius of conv

$$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Idea

let $a=0$



$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

$$f(x) = \underbrace{c_0}_{f(0)} + \underbrace{c_1}_{f'(0)} x + \underbrace{c_2}_{\frac{1}{2!}f''(0)} x^2 + \underbrace{c_3}_{\frac{1}{3!}f^{(3)}(0)} x^3 + \dots$$

Same height @ $x=0$ $\quad f(0) = c_0$

deriv (right) $= c_1 + 2c_2 x + \dots$ $\quad 3c_3x^2$

Same slope (same 1st deriv) @ $x=0$ $\quad f'(0) = c_1$

deriv (1st deriv) $= 2c_2 + 3\cdot 2\, c_3 x + 4\cdot 3\, c_4 x^2 + \dots$

Same 2nd deriv @ $x=0$ $\quad f''(0) = 2c_2 \to c_2 = \frac{1}{2} f''(0)$

deriv (2nd deriv) $= 3!\, c_3 + 4\cdot 3\cdot 2\, c_4 x + 5\cdot 4\cdot 3\, c_5 x^2 + \dots$

Same 3rd deriv @ $x=0$ $\quad f'''(0) = 3!\, c_3 \to c_3 = \frac{1}{3!} f'''(0)$

⋮ continue

about $x=0$

If $f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$

then
$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \frac{1}{3!} f'''(0)\,x^3 + \dots$$

ex $f = e^x \quad f' = e^x \quad f'' = e^x \quad f''' = e^x \quad f^{(4)} = e^x$

@ $x=0 \quad f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 1 \quad f^{(3)}(0) = 1, \dots$

so
$$e^x = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots$$
$$= 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

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check $f = (1-x)^{-1},\ f' = (1-x)^{-2},\ f'' = 2(1-x)^{-3},\ f''' = 3!(1-x)^{-4},\ f^{(4)} = 4!(1-x)^{-5}$

@ $x=0 \quad f(0) = 1,\ f'(0) = 1,\ f^{(2)}(0) = 2!,\ f^{(3)}(0) = 3!,\ f^{(4)}(0) = 4!, \dots$

So
$$\frac{1}{1-x} = f(0) + f'(0)\,x + \frac{1}{2!} f^{(2)}(0)\,x^2 + \frac{1}{3!} f^{(3)}(0)\,x^3 + \dots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Call the Maclaurin Series to be ...

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots$$

and
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$$