Math 243

11.10 Taylor Series / Maclaurin Series

about $x=a$ let $a=0$

Thm If f has a power series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $|x-a|<R$

then $c_n = \dfrac{f^{(n)}(a)}{n!}$

or $$\underbrace{f(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots}_{\text{taylor series}}$$

when $a=0$
$$\underbrace{f(x) = f(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \dots}_{\text{Maclaurin series}}$$

So if $\sin x$ does have a power series @ $x=0$ then ..

$\sin x = ?$

$$f(x) = f(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

@ 0

$f(x) = \sin(x)$ $\quad$ $f(0) = 0$

$f^{(1)}(x) = \cos(x)$ $\quad$ $f^{(1)}(0) = 1$

$f^{(2)}(x) = -\sin(x)$ $\quad$ $f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos(x)$ $\quad$ $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad$ $f^{(4)}(0) = 0$

$\vdots$ $\quad$ $\vdots$

So $\sin(x) = \underbrace{f(0)}_{0} + \underbrace{f^{(1)}(0)}_{1} x + \frac{\underbrace{f^{(2)}(0)}_{0}}{2!} x^2 + \frac{\underbrace{f^{(3)}(0)}_{-1}}{3!} x^3 + \frac{\underbrace{f^{(4)}(0)}_{0}}{4!} x^4 + \ldots$

$$\boxed{\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \frac{1}{9!} x^9 - \ldots}$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)!} x^{2n-1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

radius of conv (use ratio test)

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1} \frac{1}{(2n+3)!} x^{2n+3}}{(-1)^n \frac{1}{(2n+1)!} x^{2n+1}} \right| = \lim_{n \to \infty} \frac{(2n+1)!}{(2n+3)!} |x|^2$$

$$= \lim_{n \to \infty} \frac{1}{(2n+3)(2n+2)} x^2 = 0 \cdot x^2 = 0 < 1$$

always!

So $R = \infty$

but how can we know $\sin(x)$ has a power series?

Idea: $f(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots$

consider

$$f(x) = \underbrace{f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_k} + \underbrace{\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + \dots}_{R_k}$$

k^{th} taylor polynomial

use $f(x) = T_k + R_k \longrightarrow R_k = f(x) - T_k$

idea: if $R_k \to 0$ as $k \to \infty$

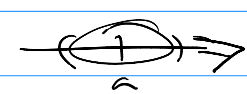
then $T_k \to f(x)$ as $k \to \infty$

and $f(x) =$ taylor series

Taylor's Inequality

b/c $R_k = \frac{1}{(k+1)!} f^{(k+1)}(a)(x-a)^{k+1} + \dots$

if $\left| f^{(k+1)}(x) \right| < M$ for $|x-a| < d$



then $|R_k| \leq \frac{M}{(k+1)!}(x-a)^{k+1}$

So back to sin(x)

$$\sin x \approx \overbrace{0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \dots + \frac{f^{(K)}(0)}{K!}x^K}^{T_K}$$

$$R_K = \frac{f^{(K+1)}(0)}{(K+1)!}x^{K+1} + \frac{f^{(K+2)}(0)}{(K+2)!}x^{K+2} + \dots$$

<u>Need</u> $\lim\limits_{K\to\infty} R_K = 0$

1st $f^{(K+1)}(x) = \begin{cases} \sin x \\ \cos x \\ -\sin x \\ -\cos x \end{cases}$ on $(\,d \quad 0 \quad d\,) \longrightarrow x$

<u>Know</u> $f^{(K+1)}(x) \le 1$

2nd So $|R_K| \le \frac{1}{(K+1)!}x^{K+1}$

3rd $\lim\limits_{K\to\infty} \frac{x^{K+1}}{(K+1)!} = 0$

So $\lim\limits_{K\to\infty} R_K = 0$ by squeeze thm.

$$\therefore \quad \boxed{\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots} \quad R = \infty$$

ex $\quad \sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} + \dots$

$\sin(0) = 0$

$\sin(0.01) =$

$\sin(0.02)$

$\sin(0.03)$

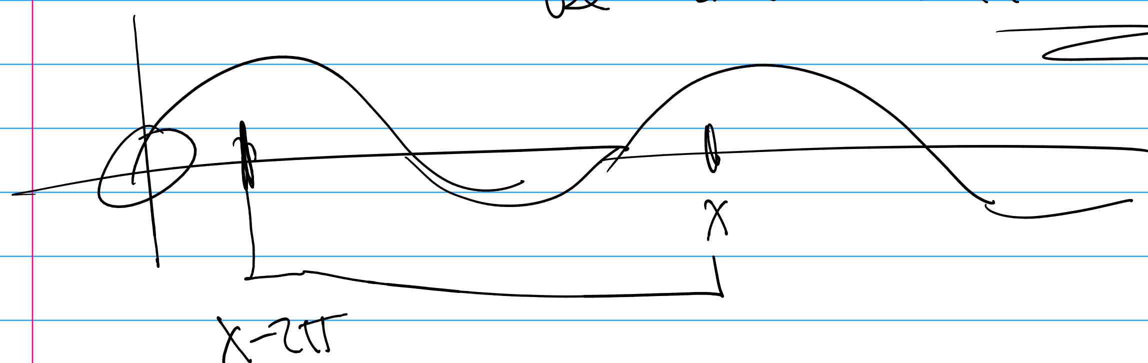
$\vdots$

$\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \dots$

$\vdots$

$\underline{\sin(1000)} = \underline{1000 - \frac{1}{3!}(1000)^3 + \frac{1}{5!}(1000)^5 -}$

use $\sin(x) = \underline{\sin(x - 2n\pi)}$



Known Maclaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{on } |x| < 1$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad \text{on } R = \infty$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad \text{on } R = \infty$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \quad \text{on } R = \infty$$
