Math 243

$\boxed{\text{Q's}}$ $\boxed{\dfrac{x}{16+x^2}}$ = power series?

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots \qquad |\square| < 1 \quad (R=1)$$

$$e^{\square} = 1 + \square + \frac{1}{2!}\square^2 + \frac{1}{3!}\square^3 + \dots \qquad R = \infty$$

$$\frac{x}{16+x^2} = x\left(\frac{1}{16+x^2}\right) = \frac{x}{16}\left(\frac{1}{1+\frac{x^2}{16}}\right) = \frac{x}{16}\left(\frac{1}{1-\frac{-x^2}{16}}\right)$$

So

$$\frac{x}{16}\left(1 + \left(\frac{-x^2}{16}\right) + \left(\frac{-x^2}{16}\right)^2 + \left(\frac{-x^2}{16}\right)^3 + \dots\right) \text{ on } \left|\frac{-x^2}{16}\right| < 1$$

$$= \frac{x}{16} - \frac{x^3}{16^2} + \frac{x^5}{16^3} - \frac{x^7}{16^4} + \dots \quad \text{on} \quad |x| < 4$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{16^n} x^{2n-1} \quad \text{on} \quad |x| < 4$$

$\int \left[\dfrac{t}{1-t^7}\right] dt$ use $\dfrac{t}{1-t^7}$ = power series?

$$\frac{t}{1-t^7} = t\left[\frac{1}{1-t^7}\right] = t\left[1 + t^7 + t^{14} + t^{21} + \dots\right] \text{ on } |t^7| < 1$$

$|t| < 1$

$$\left[\frac{1}{1-\square}\right] = \left[1 + \square + \square^2 + \square^3 + \dots\right] \text{ on } |\square| < 1$$

$$\int \frac{t}{1-t^7}\,dt = \int \left[t^1 + t^8 + t^{15} + t^{22} + \dots\right] dt \quad \text{on} \quad \underline{|t|<1}$$

$$= C + \frac{1}{2}t^2 + \frac{1}{9}t^9 + \frac{1}{16}t^{16} + \frac{1}{23}t^{23} + \dots \quad \text{on} \quad |t|<1$$

$$= C + \sum_{n=0}^{\infty} \frac{1}{7n+2} t^{7n+2} \quad \text{on} \quad |t|<1$$

<u>Taylor Series:</u> $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

let $a=0$ <u>on radius of conv. R</u>

Maclaurin $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

→ Taylor Polynomials of degree K

$$T_K(x) = \sum_{n=0}^{K} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

<u>we now have...</u> (Maclaurin)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{on} \quad R=1$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad \text{on} \quad R=\infty$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad \text{on} \quad R=\infty$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \quad \text{on} \quad R=\infty$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad \text{on} \quad R=1$$

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad \text{on} \quad R=1$$

$(1+x)^p$ $\quad$ p is any real number

$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$ or $\sqrt[3]{1+x} = (1+x)^{1/3}$

$(1+x)^p$ = power series (Maclaurin) @ $x=0$

$f(x) = (1+x)^p$ $\qquad$ $f(0) = 1$

$f^{(1)}(x) = p(1+x)^{p-1}$ $\qquad$ $f^{(1)}(0) = p$

$f^{(2)}(x) = p(p-1)(1+x)^{p-2}$ $\qquad$ $f^{(2)}(0) = p(p-1)$

$f^{(3)}(x) = p(p-1)(p-2)(1+x)^{p-3}$ $\qquad$ $f^{(3)}(0) = p(p-1)(p-2)$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}$ $\qquad$ $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$

Maclaurin

$$f(x) = f(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$\boxed{(1+x)^p = \sum_{n=0}^{\infty}\binom{p}{n}x^n} \qquad \binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$$

ex $$(1+x)^{1/2} = 1 + \tfrac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots$$

$\Rightarrow$ by ratio test $\quad R = 1$

Using Power Series

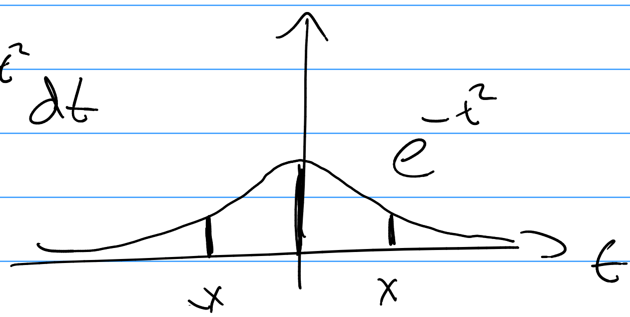
(ex) $\sin(1)$ = evaluate transcendental function

→ non-elementary functions

$\boxed{1^{st}}$ elementary function: finite number of operations, functions of $+, -, *, \div$, power, trig, exponential, log, hyperbolic, inverses $f(\,)$ and composition.

$\boxed{ex}$ $$\frac{\sin\left(e^{x^2+\sqrt{x}}\right)}{\tan^{-1}\left(\sqrt{\ln x+1}\right)} + \left(\log_3 x^2 + \cos x\right)^3$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$



not elementary

$\boxed{\text{need}}$ $$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\left[\left(\int e^{-t^2}\,dt\right)\right]\Bigg|_{t=0}^{t=x}$$

$$\int e^{-t^2}\,dt = \int\left[1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \cdots\right]dt$$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}t^{2n}\right]dt$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)\,n!}t^{2n+1}$$

So $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\, n!} t^{2n+1} \right] \Bigg|_0^x$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\, n!} x^{2n+1}$$

$$\longrightarrow = \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3 \cdot 1!} x^3 + \frac{1}{5 \cdot 2!} x^5 - \frac{1}{7 \cdot 3!} x^7 + \dots \right]$$

a $R = \infty$
