Math 243

Series: $\sum a_n = ?$ convergent? divergent?

Problem

$\underbrace{1+\underbrace{2+\underbrace{3+4}+5}+6}=21$

Commutative law $a+b=b+a$
assoc. law $(a+b)+c = a+(b+c)$

together it says ...

$\underbrace{\boxed{a_1+a_2+\ldots+a_k}}_{\text{sum}} = S$ any rearrangement of a_i is still S.

Does this work for series?

$S = a_1+a_2+a_3+\ldots$

Note: divergent series

(ex) $1-1+1-1+1-1+1-1+\ldots \to$ divergent.

try to apply commutative / assoc. laws

$1\underbrace{-1+1}_{0}\underbrace{-1+1}_{0}\underbrace{-1+1}_{0}\underbrace{-1+\ldots}_{0} = 1$

$1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$

$= -1 - 1 - 1 + \underbrace{1 - 1}_{0} + \underbrace{1 - 1}_{0} + \underbrace{1 - 1}_{0} + \dots = -3$

div series $\quad \sum a_n =$ anything with some rearrangement.

1st $\quad$ if $\sum a_n$ is absolutely convergent..

(means $\sum |a_n|$ converges)

So $\quad S = a_1 + a_2 + a_3 + \dots$

and $\quad S =$ (any rearrangement of a_i series)

2nd $\quad \sum a_n$ is conv. and $\sum |a_n|$ is divergent

and $\quad$ for any real number, r

(you can find some rearrangement of a_i) $= r$

Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = S$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\ln 2}$

(conv. alt. series)

VS $\quad \displaystyle\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad$ div. p-series ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = S$

$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \dots$

$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$

$\frac{1}{2}S = \quad \frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \frac{1}{10} + \dots$

$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \dots$

S rearranged.

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is $\sum a_n$ convergent? divergent?

11.7 **Strategy**

0 Know: - all the tests
- p-series
- geometric series
- alt. series

1 quick guess $\longrightarrow$ Does div. test apply?

"like" for large n

ex $\sum_{n=1}^{\infty} \frac{\sin(3n)}{\sqrt{n^3+2}}$ $\qquad \frac{\sin(3n)}{\sqrt{n^3+2}} \rightsquigarrow \frac{1}{n^{3/2}}$

2 Comparisons (Comparison test, limit comparison test)

3 root / ratio test

$$\sum \frac{\sin(3n)}{\sqrt{n^3+2}} \quad \text{guess } \underline{\text{conv.}}$$

$\boxed{\text{Comparison test}}$

$$\frac{\sin(3n)}{\sqrt{n^3+2}} < \frac{2}{\sqrt{n^3+2}} < \frac{2}{\sqrt{n^3}} = \frac{2}{n^{3/2}}$$

conv.

So $\sum a_n$ $\boxed{\text{conv}}$

ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\sin(3n+3)}{\sqrt{(n+1)^3+2}}}{\frac{\sin(3n)}{\sqrt{n^3+2}}} \right| = ?$$

Ex

$$\sum_{n=1}^{\infty} \frac{\sqrt[4]{n^7+1}}{\sqrt{n^2-1}}$$

$\underline{\text{check}}$ "like" $\frac{\sqrt[4]{n^7+1}}{\sqrt{n^2-1}} \sim \frac{n^{7/4}}{n^{2/2}} = n^{3/4}$ as $n\to\infty$

$$\lim_{n\to\infty} \frac{\sqrt[4]{n^7+1} \cdot \frac{1}{n}}{\sqrt{n^2-1} \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[4]{n^3+\frac{1}{n^4}}}{\sqrt{1-\frac{1}{n^2}}} = \boxed{\infty} \neq 0$$

div

{ guess div and use div. test

$$\lim(\sim) = \lim \sin(n) \neq 0$$

11.8 back to the conv. geometric series ...

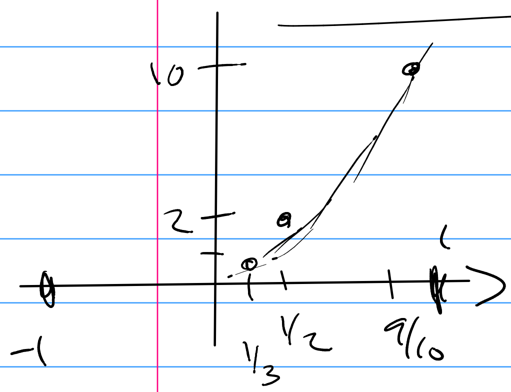
$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \quad \text{when } \underline{|x| < 1}$$

So if $x = \frac{1}{2}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1-\frac{1}{2}} = \boxed{2}$$

see $\frac{1}{2} \to 2$, $\frac{1}{3} \to 3/2$, $\frac{9}{10} \to 10$

So $1 + x + x^2 + x^3 + \dots$ conv. for $|x| < 1$



so $f(x) = 1 + x + x^2 + \dots$ $|x| < 1$

natural domain of $f(x)$

and $f(x) = \dfrac{1}{1-x}$ only on $|x| < 1$

Note: $\dfrac{x+1}{x^2-1} = \dfrac{\cancel{(x+1)}}{\cancel{(x+1)}(x-1)} = \dfrac{1}{x-1}$, $x \neq -1$

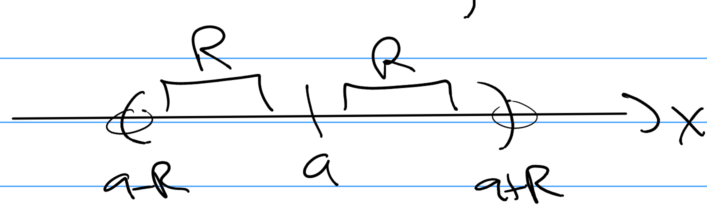
domain $x \neq 1$
$x \neq -1$

So $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \dfrac{1}{1-x}$ on $|x| < 1$

generalize

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Power series about $x = a$, radius of conv R



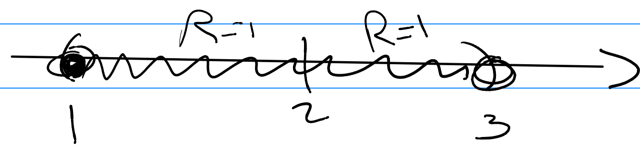
ex $\sum_{n=0}^{\infty} \frac{1}{n}(x-2)^n$

test for abs. conv. $\lim_{n\to\infty} \left| \dfrac{\frac{1}{n+1}(x-2)^{n+1}}{\frac{1}{n}(x-2)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x-2|$

$= 1 \cdot |x-2|$ so $L = |x-2|$

abs conv when $L < 1 \to \underline{|x-2| < 1}$

R $\boxed{R=1}$



check endpts:

$x=1$

$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$

conv

$x=3$

$\sum_{n=0}^{\infty} \frac{1}{n}$

div.