Math 243

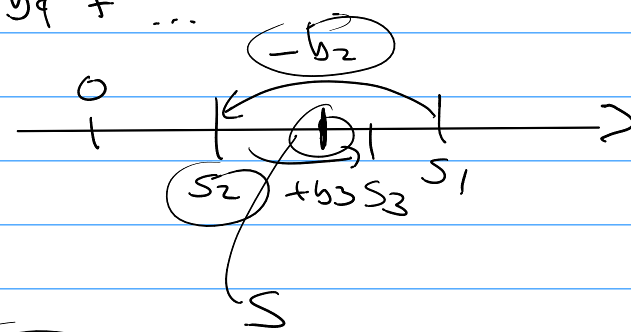
Q's $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1}\frac{1}{n}$

$n=1$ $n=2$ $\underbrace{(-1)^{n-1}\frac{1}{n}}_{a_n}$

over vs under estimates?

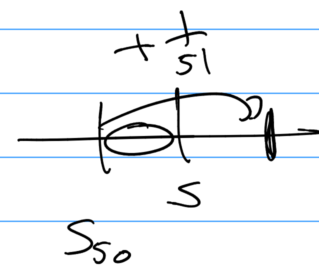
Alt. Series $0 + \underbrace{b_1}_{} - b_2 + b_3 - b_4 + \dots$

S_1, S_2, S_3



$S_{50} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots - \frac{1}{50}$

underestimate means $S_{50} < S$



error $< b_{51} = \frac{1}{51}$

11.9 Power Series as functions

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

on $|x-a| < R$

1st equality of functions

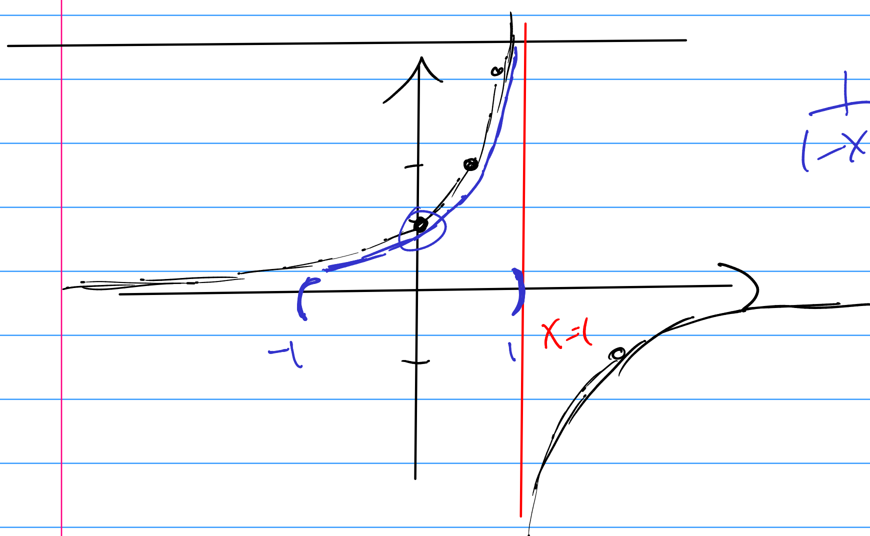
ex $\sqrt{x^2} = |x|$

ex $\frac{x+3}{x^2-9} = \frac{1}{x-3}, x \neq -3$

2nd $\quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ when $|x| < 1$

ex) $1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{1-\frac{1}{2}} = \boxed{2}$



$\frac{1}{1-x}$

$\frac{1}{1-x} = 1 + x + x^2 + \ldots \quad |x|<1$

So we have at least this equality of functions..

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad |x|<1$$

on this restricted domain

Can we make others?

things we can do..

1. Substitution

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \ldots \quad |\square| < 1$$

ex) let $\square = 1-x \rightarrow \frac{1}{x} = 1 + (1-x) + (1-x)^2 + \ldots \quad |1-x|<1$

So $\frac{1}{x} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + \dots$ $\quad |1-x| < 1$

$|x-1| < 1$

Let $\square = -x^2$ $\quad \frac{1}{1-\square} = 1 + \square + \square^2 + \dots$ $\quad |\square| < 1$

So $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$ $\quad |-x^2| < 1$

$|x^2| < 1$

$x^2 < 1$

$x^2 - 1 < 0$

$(x+1)(x-1) < 0$

$\boxed{|x| < 1}$

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power series for $\frac{x^3}{2+x} = \frac{x^3}{2}\left(\frac{1}{1 + x/2}\right) = \frac{x^3}{2}\left(\frac{1}{1 - \frac{-x}{2}}\right)$

know $\left(\frac{1}{1-\square}\right) = 1 + \square + \square^2 + \dots$ $\quad |\square| < 1$

Let $\square = -x/2$

So $\frac{x^3}{2+x} = \frac{x^3}{2}\left(1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \left(-\frac{x}{2}\right)^3 + \dots\right)$ on $\left|\frac{-x}{2}\right| < 1$

$\frac{x^3}{2+x} = \frac{x^3}{2} - \frac{x^4}{2^2} + \frac{x^5}{2^3} - \frac{x^6}{2^4} + \dots$ on $|x| < 2$

② Derivatives

if $f_1(x) = f_2(x)$

then $\frac{d}{dx}[f_1(x)] = \frac{d}{dx}[f_2(x)]$

③ Integrals

$f_1(x) = f_2(x)$

$\int f_1\,dx = \int f_2\,dx$

$\rightarrow$ $\underline{|x-a| < R}$ Radius of conv. stays the same

So $\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots \quad |\square| < 1$

Sub $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \quad |x| < 1$

Deriv $\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{d}{dx}\left[(1+x)^{-1}\right] = \frac{-1}{(1+x)^2}$

$\frac{d}{dx}\left[1 - x + x^2 - x^3 + x^4 - x^5 + \dots\right] = -1 + 2x - 3x^2 + 4x^3 - \dots$ on $|x| < 1$

So $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$ on $|x| < 1$

Integration $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$ on $|x| < 1$

$\int \frac{1}{1+x}\,dx = \ln|1+x| + C_1$

$\int \left[1 - x + x^2 - x^3 + x^4 - \dots\right] dx = C_2 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

So $\ln|1+x| = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ on $|x| < 1$

Put $x=0 \rightarrow$ find $C = 0$

$\therefore \ln(1+x) = x - \frac{1}{2}x + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ on $|x| < 1$

(ex) $\tan^{-1}(x)$ as a power series

Know: $\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$

make a power series for this.

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one other

① $f' = f$ $\rightarrow$ know $f(x) = Ke^x$

② consider $f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$

(use ratio test to show $R = \infty$)

③ $\rightarrow f'(x) = 0 + \underbrace{1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots}_{f(x)}$

So $f' = f$ $\rightarrow$ therefore $f(x) = Ke^x$

$\therefore Ke^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$ on $R = \infty$

let $x = 0 \rightarrow K = 1$

at last $\boxed{e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad \text{on } R = \infty}$

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

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ex $\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$ $\qquad \frac{1}{1-\square} = 1 + \square + \square^2 + \dots$

at $\square = -x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots \quad \text{on } |x| < 1$$

$$\tan^{-1}(x) = \int [1 - x^2 + x^4 - x^6 + \dots]\,dx$$

$$= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

@ $x = 0 \to 0 = C$

$$\therefore \tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad \text{on } |x| < 1$$

Note: with much work it also converges @ $x = 1$

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$
