

Math 243

Power Series :-

$$\textcircled{1} \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{on } |x| < 1 \quad (R=1)$$

Let $x = ?$ for several substitutions (\rightarrow) Deriv (\rightarrow) integration

$$\textcircled{2} \quad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad \text{on } |x| < 1 \quad (R=1)$$

$$\textcircled{3} \quad \tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad \text{on } |x| < 1 \quad (R=1)$$

$$\textcircled{4} \quad e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad \text{on } R = \infty$$

Idea: Given any $f(x)$, find a power series that it is equal to and its radius of convergence ..

About $x=0$

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots = \sum_{n=0}^{\infty} c_n x^n$$

About $x=a$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

want $f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$

assume: know $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, \dots

$$f(0) = (\text{Power Series at } x=0) = C_0$$

also $f'(x) = \frac{d}{dx} [\text{Power Series}] = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \dots$

$$\rightarrow f'(0) = C_1$$

also $f''(x) = 2C_2 + 3 \cdot 2C_3 x + 4 \cdot 3 \cdot C_4 x^2 + \dots$

$$f''(0) = 2C_2 \rightarrow C_2 = \frac{1}{2} f''(0)$$

also $f'''(x) = \cancel{3 \cdot 2} \cdot C_3 + 4 \cdot 3 \cdot 2 C_4 x + \dots$

$$f'''(0) = 3! C_3 \rightarrow C_3 = \frac{1}{3!} f'''(0)$$

Continue $C_n = \frac{1}{n!} f^{(n)}(0)$

OK

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

$$R = ?$$

check

$$f(x) = e^x \quad f(0) = 1$$

$$f'(x) = e^x \quad f'(0) = 1$$

$$f''(x) = e^x \quad f''(0) = 1$$

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

use ratio test

to find this

$$R =$$

So far $\Rightarrow f$ $f(x)$ has a power series, it is the above.

\checkmark Does $f(x)$ have a power series?

at $x=a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Values:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \quad \text{Taylor Series}$$

$$(a=0) \quad f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \quad \text{Maclaurin Series}$$

$$\boxed{\Rightarrow} f(x) = \left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k \right] + \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} + \dots$$

$$e^x = \left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \right] + \left[\frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots \right]$$

↗ 3rd degree poly rest

$$\text{Now } f(x) - T_k = \text{Error} = R_k = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} + \dots$$

Say $f(x) = \text{Taylor Series}$

$$f \quad \lim_{k \rightarrow \infty} R_k = 0$$

Taylor's Inequality

$$R_k = \frac{f^{(k+1)}(a)}{(k+1)!} (x-a)^{k+1} + \frac{f^{(k+2)}(a)}{(k+2)!} (x-a)^{k+2} + \dots$$

$$f \left| \frac{f^{(k+1)}}{f(x)} \right| \leq M \quad \text{on} \quad |x-a| < d \quad f(x)$$

then $|R_k| \leq \frac{M}{(k+1)!} |x-a|^{k+1}$



Ex $f(x) = \sin x \quad f(0) = 0$
function $f'(x) = \cos x \quad f'(0) = 1$
 $f''(x) = -\sin x \quad f''(0) = 0$
 $f'''(x) = -\cos x \quad f'''(0) = -1$
 $f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$
 \vdots

Now: $\left| \frac{f^{(k+1)}}{f(x)} \right| = \begin{cases} \sin x \\ \cos x \\ -\sin x \\ -\cos x \end{cases} \leq 1$

$$\sin x = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{(-1)}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \dots$$

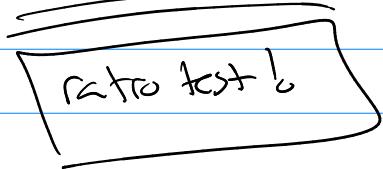
$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
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check existence $\lim_{k \rightarrow \infty} R_k = 0$

have $|R_k| \leq \frac{1}{(k+1)!} |x-a|^{k+1}$

$\underset{k \rightarrow \infty}{\lim} \frac{|x-a|^{k+1}}{(k+1)!} = 0$

Radius of conv?



$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)!} (x)^2$$

$$= \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} |x|^2 = \lim_{n \rightarrow \infty} \left(\frac{1}{(2n+3)(2n+2)} \right) x^2$$

$$= 0 \cdot x^2 = 0 \quad \text{always} \quad R = \infty$$

So

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \quad R = \infty$$

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \quad R = \infty$$

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots \quad R = \infty$$

Note: $e^{ix} = \cos x + i \sin x$

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i \cdot 0$$

So $e^{i\pi} + 1 = 0$

another $(1+x)^p$ p is any real

why? $\frac{1}{1+x} = (1+x)^{-1}$ = Power series.

$$f(x) = (1+x)^p$$

① $f(0) = 1$

$$f'(x) = p(1+x)^{p-1}$$

$x \approx 0$ $f'(0) = p$

$$f''(x) = p(p-1)(1+x)^{p-2}$$

$$f''(0) = p(p-1)$$

{So}

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \frac{p(p-1)(p-2)(p-3)}{4!} x^4 + \dots$$

$$R = 1$$

{So}

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-\frac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} x^3 + \dots$$

Notation:

$$\frac{p(p-1)(p-2)\dots(p-n+1)}{n!} = \binom{p}{n}$$

also polynomial $+, -, \times, \div$ can be used a these series.

$$④ \sin x + x \cos x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots$$

$$+ \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \right)$$

$$= 2x - \left(\frac{1}{3!} + \frac{1}{5!} \right) x^3 + \left(\frac{1}{5!} + \frac{1}{7!} \right) x^5 - \left(\frac{1}{7!} + \frac{1}{9!} \right) x^7 + \dots$$

Defn: $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (Maclaurin Series)

(why?)

① Math evaluation.

(Ex) $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots$

$$\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots$$

$$\rightarrow \text{do } \sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!}$$

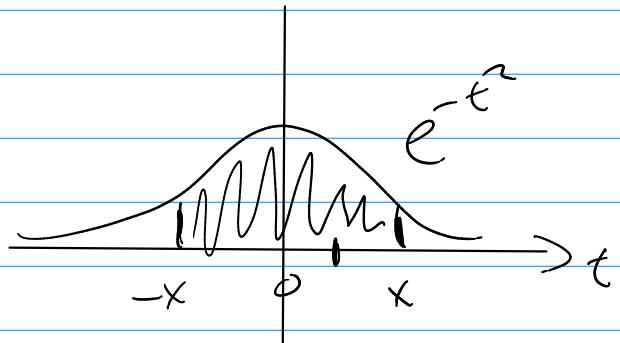
② Polynomial approximations..

$$f(x) \approx T_k$$

(Ex) $\frac{1}{1+x} \approx 1 - \frac{1}{2}x$ near $x=0$ linear approx.

(Ex) $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

(Ex) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$



$$\int e^{-t^2} dt = \boxed{\text{not elementary}}$$

Density function:

(Ex) $\frac{\sin(\ln \sqrt{x^2+1})}{\tan^{-1}(x^3) - \sqrt{x+3}} - \cos(\bar{x}-2)$

$\text{erf}(x) = \text{Power Series}$

$$e^{-t^2} = 1 - t^2 + \frac{1}{2!} t^4 + \frac{1}{3!} t^6 + \dots$$

$$\int (e^{-t^2}) dt = \int \left(1 - t^2 + \frac{1}{2!} t^4 - \frac{1}{3!} t^6 + \dots \right) dt$$

$$= t - \frac{1}{3} t^3 + \frac{1}{5 \cdot 2!} t^5 - \frac{1}{7 \cdot 3!} t^7 + \dots$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3} x^3 + \frac{1}{5 \cdot 2!} x^5 - \frac{1}{7 \cdot 3!} x^7 + \dots \right]$$