Math 322

Relation on Set A ; Subset of $A \times A$

Notation: $R : A \to A$ $(a,b) \in R$, aRb

Representing them

① Set Notation: a) list the ordered pairs
b) set builder

Rule of Logic

$$R = \{(a,b) \mid a \in A, b \in A, \text{and } P(a,b)\}$$

→ how many possible relations can exist?

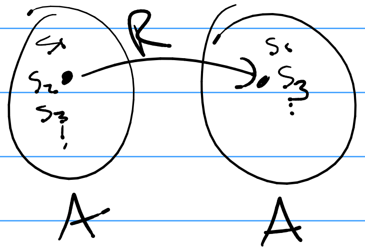
R is a subset of $A \times A$

↳ take 0 pairs → $\emptyset$
take 1 pair
take 2 pairs
⋮
take all pairs → $A \times A$

$$|P(A \times A)| = 2^{|A|^2}$$

② Visualize

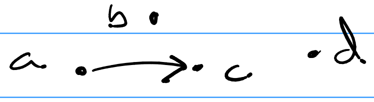
a) Arrow diagram



$(s_2,s_3) \in R$

b) directed graph : Digraph

A : set of vertices (non-empty)
E : set of edges (ordered pairs)



$(a,c) \in R$

directed graph $a \bullet \longrightarrow b$ $(a,b) \in R$

initial vertex terminal vertex directed edge

3) Numeric (bit) representations

a) tally table $(a_2,a_1) \in R$

b) Zero-one bit matrix

	R	a_1	a_2	--	a_n
		A			
A	a_1				
	a_2	X			
	⋮				
	a_n				

$$M_R = [a_{ij}]$$

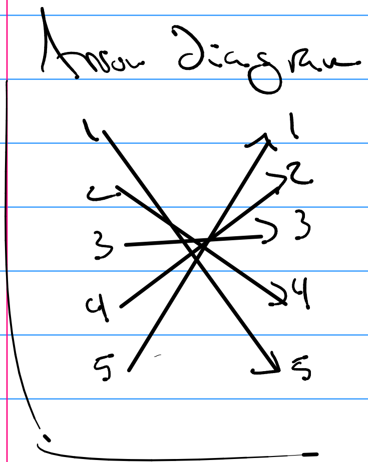
$a_{ij} = 1$ if $(a_i,a_j) \in R$

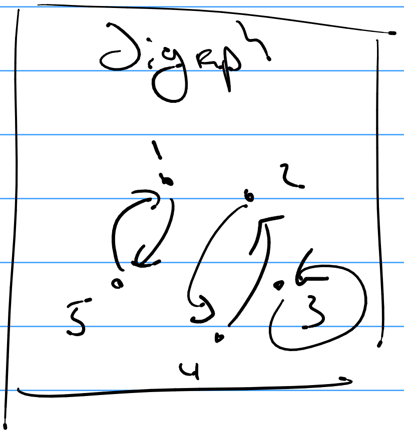
$a_{ij} = 0$ if $(a_i,a_j) \notin R$

(ex) $A = \{1,2,3,4,5\}$

$R = \{(a,b) \mid \underline{a+b} = 6\}$

$R = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$





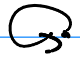
$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

any relations we can label "special"?

0) **Functions.**

1) **Reflexive:** $\forall a\,(aRa)$

digraph



Matrix

$$\begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix}$$

2) **Irreflexive:** $\forall a\,(a \not R a)$

digraph

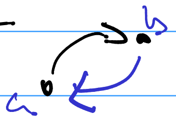
no loops

Matrix

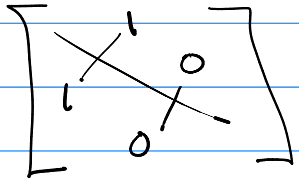
$$\begin{bmatrix}0 & & \\ & \ddots & \\ & & 0\end{bmatrix}$$

3) **Symmetric:** $\forall a \forall b\,(aRb \to bRa)$

digraph



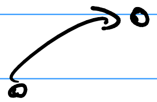
Matrix



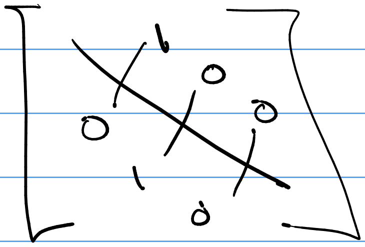
4) Antisymmetric $\forall a \forall b\,(aRb \wedge bRa \to a=b)$

$\equiv \forall a \forall b\,(a \neq b \to a \not R b \vee b \not R a)$

digraph



Matrix

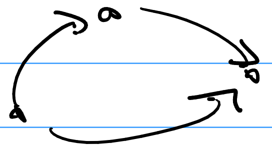


5) Asymmetric $\forall a \forall b\,(aRb \to b \not R a)$

↳ irreflexive <u>and</u> antisymmetric

(6) **Transitive:** $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

digraph

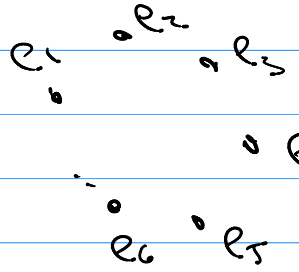


matrix

?

How to know transitive? Hard!

Ex Properties of $R = \{ \}$



$M_R = [0]$

reflexive? No

irreflexive? Yes

Sym? Yes

antisym? yes

asym? yes

trans.? yes

How to check if R on set A has a property?

① reflexive
⋮
⑤ Asymmetry } all easy to check.

(6.) Transitive? $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

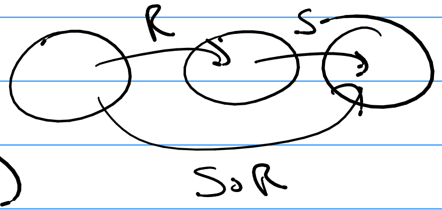
Thm R is transitive iff $\underline{R^n} \subseteq R$, $n = 1, 2, 3, \ldots$

Operations ① Set ops

$R_1 \cup R_2$ $\quad M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$

$R_1 \cap R_2$ $\quad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$

② Composition: $S \circ R$



Functional notation: $S(R(a))$

$$M_{S \circ R} = M_R \odot M_S$$

Powers: Basis: $R^1 = R$

Inductive: $R^n = \underline{R^{n-1}} \circ R$ $\quad n = 2, 3, 4, \ldots$

Back to

thm $\left(R \text{ is transitive} \leftrightarrow R^n \subseteq R, \; n = 1, 2, 3, \ldots \right) \equiv T$

Notes:

case1 left → right $\quad$ case2 right → left

① transitive? $\forall a \forall b \forall c \left(aRb \wedge bRc \rightarrow aRc \right)$

assume.

② qRs mean $(q,s) \in R$ $\quad q \xrightarrow{R} s$

③ Saw 1, 2, 3, ... ok.. we are probably going to use inductio

④ $\boxed{R^n \subseteq R}$ So $S \subseteq T$

all e $(e \in R^n \to e \in \boxed{R})$ all e $(e \in S \to e \in T)$

Note e is an ordered pair

so $((a,b) \in R^n \to (a,b) \in R)$

Now we have..

Th^m $(aRb \wedge bRc \to aRc)$ iff $(aR^n b \to aRb)$ $n=1,2,3..$

Pf **case 1** left → right

(try direct) assume R is transitive, show $(aR^n b \to aRb)$ for all $n=1,2,3,..$ by Induction

Basis: $(n=1)$

$aRb \to aRb$ **true**

Inductive: assume: k^{th} $aR^k b \to aRb$

show $(k+1)^{st}$ $aR^{k+1} b \to aRb$

$aR^{k+1} b \equiv a(R^k \circ R) b \equiv aRc \wedge cR^k b$

$\overset{IH}{\equiv} aRc \wedge cRb \equiv aRb$ **true**

(still need to finish right → left)