Math 322

$\boxed{Q's}$ 9.1 #35 $A = \mathbb{R}$

$R_2 = \{(a,b) \mid a \geq b\}$

$R_3 = \{(a,b) \mid a < b\}$

$R_4 = \{(a,b) \mid a \leq b\}$

$R_5 = \{(a,b) \mid a = b\}$

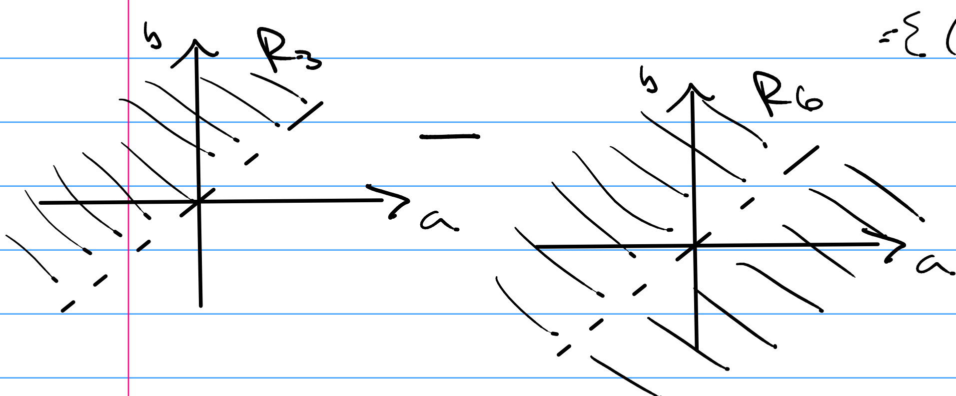
$R_6 = \{(a,b) \mid a \neq b\}$

ex) $R_3 - R_6 = \{(a,b) \mid (a,b) \in R_3 \wedge (a,b) \notin R_6\}$

"clean"

$R_3 - R_6 = \{(a,b) \mid (a<b) \wedge \neg(a \neq b)\}$

$= \{(a,b) \mid (a<b) \wedge (a=b)\}$



$= \emptyset$

$M_{R^n} = M_R \odot M_{R^{n-1}}$

$\boxed{th^m}$ R is transitive iff $R^n \subseteq R$ $n = 1, 2, 3, \ldots$

$\to$ restate

$(sRt \wedge tRq \to sRq)$ iff $\square R^n \triangle \to \square R \triangle$ $n = 1, 2, 3, \ldots$

$\boxed{Pf}$ (left $\to$ right case)

assume: ① R is transitive

② Use induction to show right

a) Basis: show: $aR^1b \to aRb$ true

c) Inductive: (I. H.) assume $\boxed{aR^kb \to aRb}$

Show: $aR^{k+1}b \to aRb$

$$\boxed{a R^{k+1} b} \equiv a R^{k} \circ R\, b \equiv \boxed{a R c \wedge c R^{k} b}$$

$$\underset{\text{I.H.}}{\rightarrow} a R c \wedge c R b \Rightarrow \boxed{a R b} \quad \text{true.}$$

(right → left case)

$R^n \subseteq R \quad n = 1, 2, \ldots \quad \rightarrow \quad R$ is transitive

rewrite

$$\left((e_1 R^n e_3) \rightarrow e_1 R e_3 \right) \rightarrow \left((e_1 R e_2 \wedge e_2 R e_3) \rightarrow e_1 R e_3 \right)$$

assume $\quad e_1 R^n e_3 \rightarrow e_1 R e_3$ for all $n = 1, 2, 3, \ldots$

by univ. instation let $n=2$ $\left(\boxed{e_1 R^2 e_3} \rightarrow e_1 R e_3 \right)$ is true

def of power

$$\equiv \left(\boxed{e_1 R \circ R e_3} \rightarrow e_1 R e_3 \right)$$

$$\equiv \left(\boxed{e_1 R e_2 \wedge e_2 R e_3} \rightarrow e_1 R e_3 \right) \equiv \text{trans. is true}$$

def. of composition

Def.: Power

basis $\boxed{R^1 = R}$

induction rule. $\boxed{R^n = R^{n-1} \circ R}$

ref, irref, sym, antisym, asym, trans

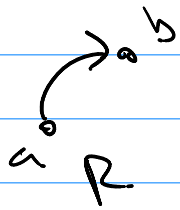
9.4 Closures

Q) What if a relation does not have property P then can I make a new relation (based on the old) that does have property P.

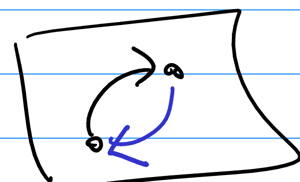
closure of R with respect to property P

goal: to do as little as possible to R.

Idea:



not symmetric

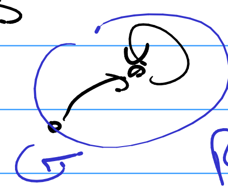


New relation is Closure of R with respect to symmetry

Properties:

OK ① reflexive (if not reflexive it's missing loops)

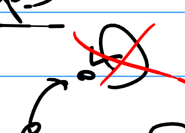
→ union loops



$R \subseteq$ New one

no ② Irreflexive (if not it has loops)

→ take out loops



$R \not\subseteq$ new one

Closures

1. reflexive
2. symmetric
3. transitive

Reflexive Closure:

Def: $\Delta = \{ (a,a) \mid aRa \}$

$$M_\Delta = I$$

$$\boxed{\begin{array}{l} \rightarrow \text{Ref. Closure} = R \cup \Delta \\ M_{\text{ref. closure}} = M_R \vee I \end{array}}$$

Symmetric Closure

Idea $a \rightleftarrows b$

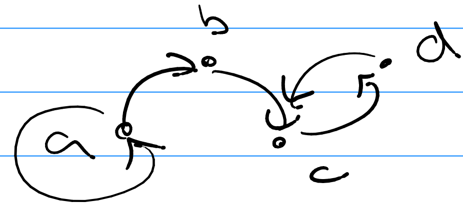
Def: $R^{-1} = \{ (b,a) \mid aRb \}$

$$M_{R^{-1}} = M_R^T$$

$$\boxed{\begin{array}{l} \text{Sym Closure} = R \cup R^{-1} \\ M_{\text{sym. closure}} = M_R \vee M_R^T \end{array}}$$

Transitive Closure

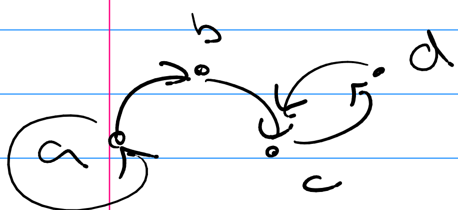
Background



a) Path from x_0 to x_n is a seq of n-edges in the graph G.

b) length is number of edges

c) cycle / circuit $\quad x_0 = x_n \quad n \geq 1$



a to c on path ...

(a,a), (a,b), (b,c), (c,d), (d,c)

length = 5

Notation: a, a, b, c, d, c $\rightarrow$ length = 5

Problem: a, a, b, d, c, d, c not a path

Thm $\quad aR^n b$ **iff** a path of length n from a to b.

$(a,b) \in R^n$

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back to $\quad R$ is trans iff $\quad R^n \subseteq R \quad n = 1, 2, 3 \ldots$

Consider: $R^* = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots$

connectivity relation, $R^* = \bigcup_{n=1}^{\infty} R^n$

$$\rightarrow \boxed{M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots}$$

Th^m: R^* is the transitive closure of R.

proof in textbook

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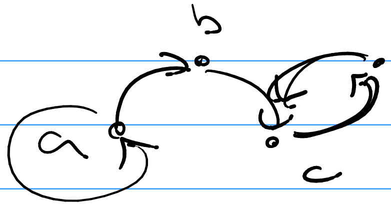
but $M_{R^*} = M_R \vee M_R^{[2]} \vee \ldots \leftarrow$ infinite!

Help: Path uniting using vertices.

ex 

$|A| = 4$

a, a, b, c, d, c, d

a, b, $\boxed{c, d, c}$

↑ Circuit (b/c pigeonhole principle)

a, b, c

?, ?, ~~?, ?, ?~~ → just cut it out.

Thm R on set A, $|A| = n$

if you have a path of length ≥ 1 from a to b

then there is a path of length not exceeding n.
And if $a \neq b$ a path of length not exceeding $n-1$.

$\rightarrow$ $\boxed{M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}}$ trans. closure

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Find M_{R^*} using Warshall's Algor.