Math 322

$\boxed{10.4}$ Paths, Isomorphism (still), # of paths using A_G^p

Connectivity

Def path: seq of edges $e_1, e_2, .., e_n$

length(path) $= n$ (# of edges)

undirected $e_i = \{v_{i-1}, v_i\}$

directed $e_i = (v_{i-1}, v_i)$

if no confusion on which edge to take, just list vertices

$v_0, v_1, v_2, .., v_n$

edge

terms: simple path: no repeated edge

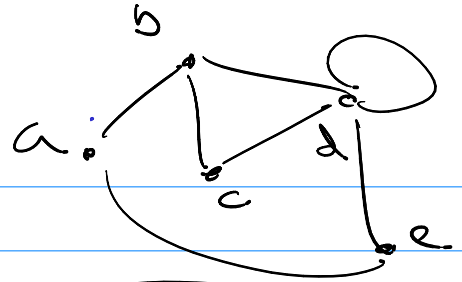
circuit: $v_0 = v_n$ $(n \geq 1)$

Counting paths from v_i to v_j of length n

thm $A_G^n = \{a_{ij}\}$

a_{ij} = # of paths from v_i to v_j of length n.

ex



$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A^2
ans = 2 paths of length 2 from a to a

$$\begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 4 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{bmatrix}$$

2 paths of length 2 from a to d.

A^3
ans = 1, 2

$A^2 \cdot A$

$$\begin{matrix} 0 & 5 & 2 & 3 & 4 \\ 5 & 3 & 5 & 8 & 2 \\ 2 & 5 & 3 & 6 & 3 \\ 3 & 8 & 6 & 9 & 6 \\ 4 & 2 & 3 & 6 & 1 \end{matrix}$$

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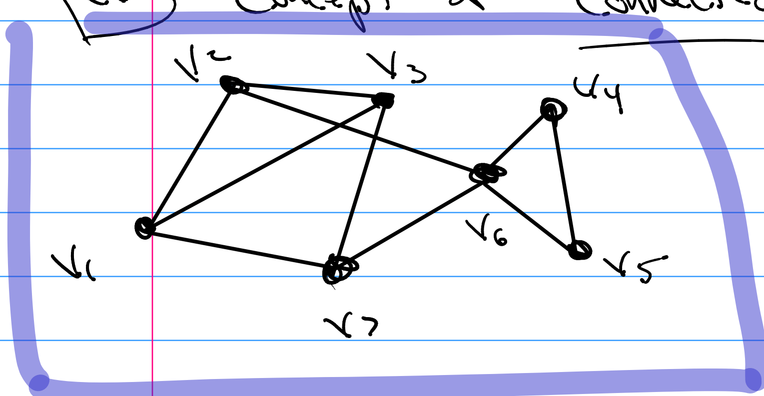
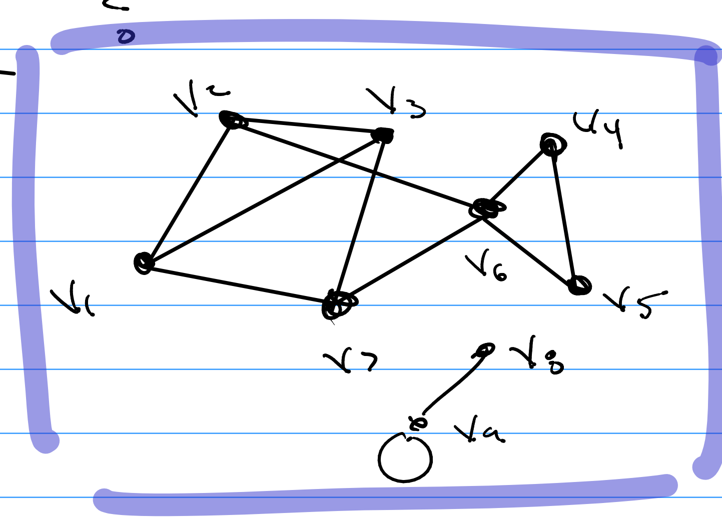
ex Concept of "connected"



G_1

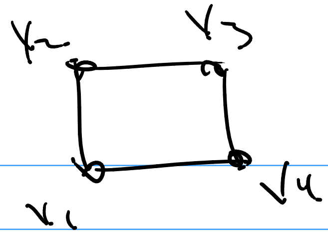
is connected



G_2

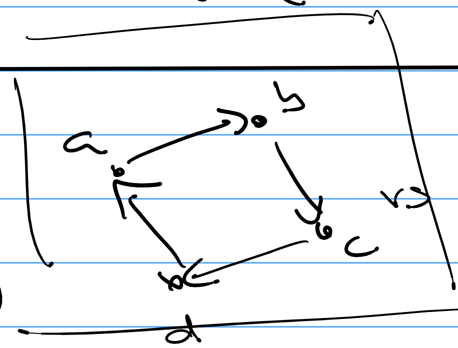
is not connected

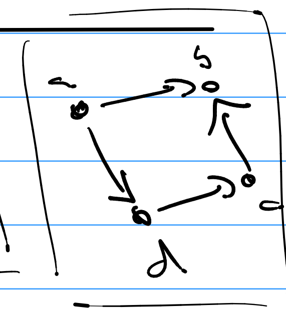
Def G is connected if between any v_i, v_j there exists some path. (undirected)



$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Connected if trans. closure has no zeros.





Directed Graph

① G is strongly connected

if there is path to and from any two vertices

② G is weakly connected

if ignoring direction the underlying undirected graph is connected.

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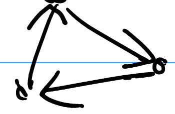
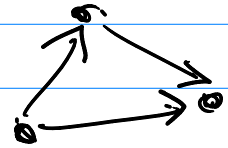
given any directed graph we have 3 possible answers

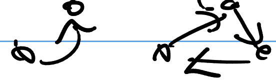
① G is strongly and weakly connected.



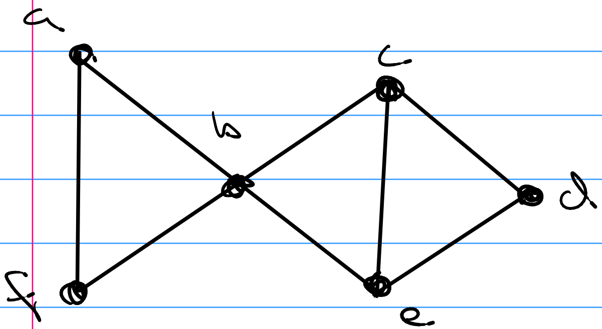
② G is not strongly and is weakly connected



③ G is not strongly and not weakly connected



is there a metric for "quality" of connectedness?



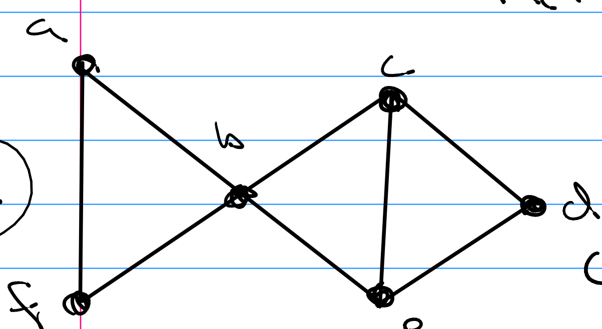
→ connected components

largest sets of a graph that are connected.

Vertex cut = set of vertices that when removed (with corresp. edges) inc. # of connected components

edge cut = set of edges that when removed inc. # of connected components.

G_1



ex vertex cuts

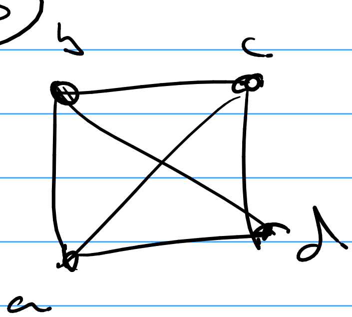
{b}, {b, e}, {b, c}

{e, c}, {e, c, b}

etc.

cut vertex

ex K_4

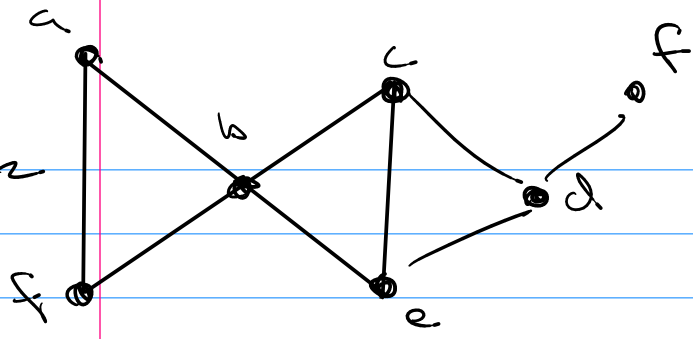


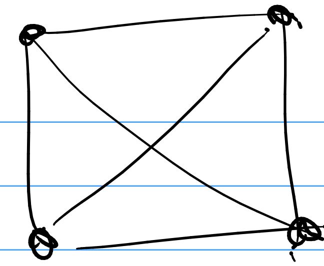
Def $K(G) = |$smallest vertex cut$|$

ex $K(G_1) = 1$

$K(K_4) = 3$ (by def)

$\boxed{K(G) \text{ is the vertex connectivity}}$

$G_2 =$ 

vs K_4 

→ remove edges

edge cut = set of edges to remove to inc. connected components

(ex) on G_1 $\{\{c,d\}, \{d,e\}\}$ an edge cut

$\{\{d,f\}\}$ an edge cut

def $\lambda(G)$ = minimal number of edges to remove and inc. connect. comp.

(ex) $\lambda(G_2) = 1$ $\lambda(K_4) = 3$

Thm $\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$

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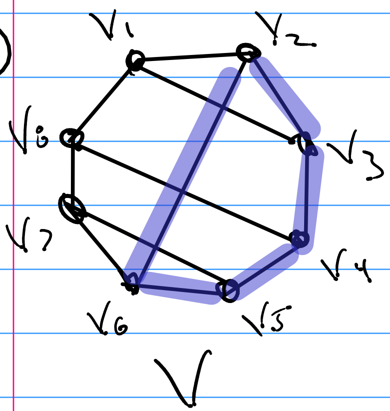
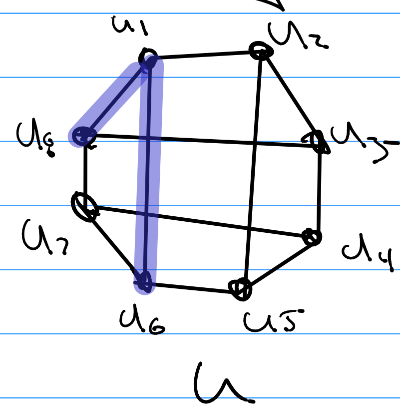
directed

→ strongly connected components.

Application | Isomorphisms

(ex)





isomorphic?

no? → find broken invariant

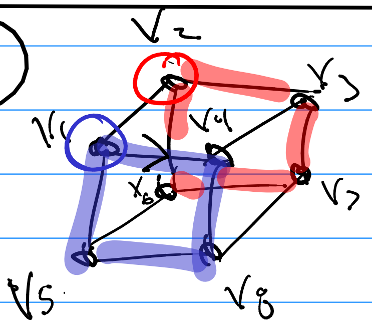
yes? → find the bijection

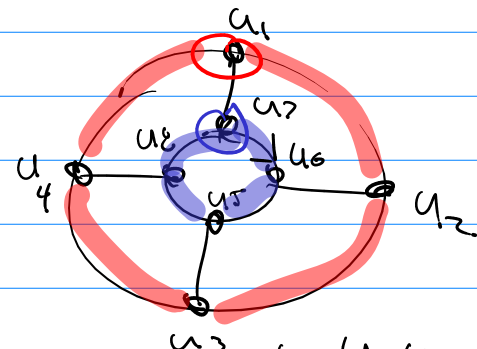
$v_i \to u_i$

$A_V = A_U$

no, b/c V has a simple circuit of length 5, U does not

(ex)





bijection f

$v_1 \to u_7$

$v_2 \to u_1$

$v_3 \to u_2$

$v_4 \to u_6$

$v_5 \to u_8$

$v_6 \to u_4$

$v_7 \to u_3$

$v_8 \to u_5$

$$A_V = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \end{array} \begin{array}{c} v_1 \; v_2 \; v_3 \cdots \\ \begin{bmatrix} \\ ? \\ \\ \end{bmatrix} \end{array}$$

$$A_U = \begin{array}{c} \\ u_7 \\ u_1 \\ u_2 \\ u_6 \\ u_8 \\ u_4 \\ u_3 \\ u_5 \end{array} \begin{array}{c} u_7 \; u_1 \; u_2 \cdots \\ \begin{bmatrix} \\ \\ ? \\ \\ \\ \end{bmatrix} \end{array}$$
