Math 322

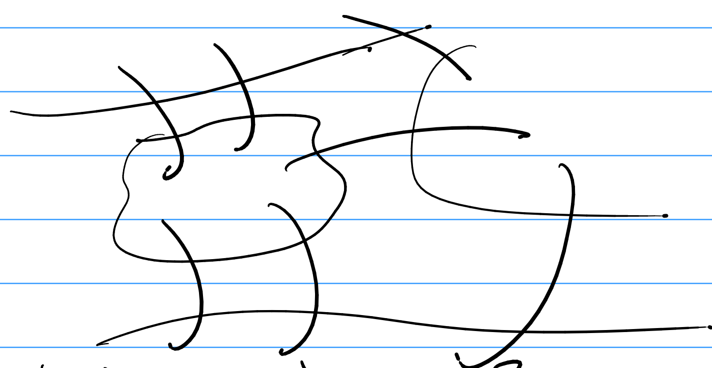
Q5 | Today: 10.5 Euler / Hamilton Paths and Circuits
(+) 10.6 shortest path problems

Thursday: 10.6 Shortest path algorithms
(+) review for Exam

Next Tues → $\boxed{\text{Exam 2}}$

10.5 Euler and Hamilton
- looking for simple paths or simple circuits

Idea



Königsberg Bridge Problem

① Visit all bridges exactly once?

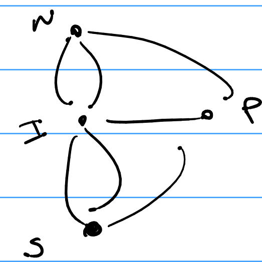
② Start and end at same place and visit all bridges exactly once?

circuit

use all edges

simple

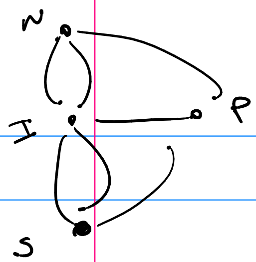
→ as a graph



edge = bridge
vertex = land

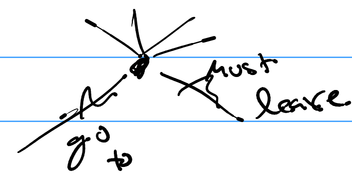
② **Def:** Euler circuit. Simple circuit that uses every edge of the graph.

① **Def:** Euler path. Simple path (not a circuit) that uses every edge of the graph.



Q's does this have an Euler path or circuit?

Note: if you do have an Euler circuit then any vertex would have

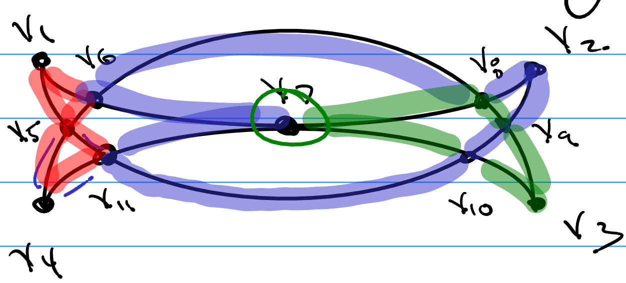


look for --

Euler Circuit $\equiv$ ("simple" statement about properties of the graph)

Thm $G = (V, E)$ has an Euler Circuit iff $\forall v \in V$ $\deg(v)$ is even

ex



v_1, v_5, v_6, v_1

v_5, v_4, v_{11}, v_5

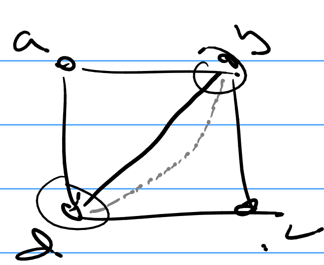
$v_1, v_5, v_4, v_{11}, v_5, v_6, v_1$

$v_{11}, v_7, v_6, v_8, v_2, v_9, v_{10}, v_{11}$

So $v_1, v_5, v_4, v_{11}, v_7, v_6, v_8, v_2, v_9, v_{10}, v_{11}, v_5, v_6, v_1$

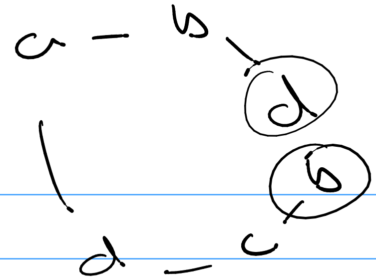
$v_7, v_8, v_9, v_3, v_{10}, v_7$

Euler Path



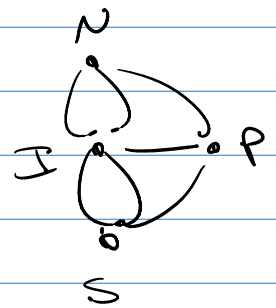
with the imaginary edge → Euler circuit

a, b, d, b, c, d, a



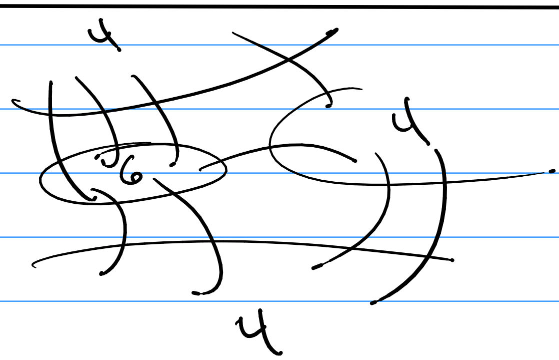
Thm $G = (V, E)$ has an Euler Path (not circuit) iff $\deg(v)$ for exactly two are odd.

Back to

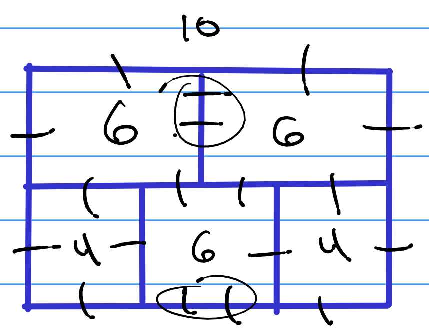


$\deg(N) = 3$
$\deg(P) = 3$
$\deg(S) = 3$
$\deg(I) = 5$

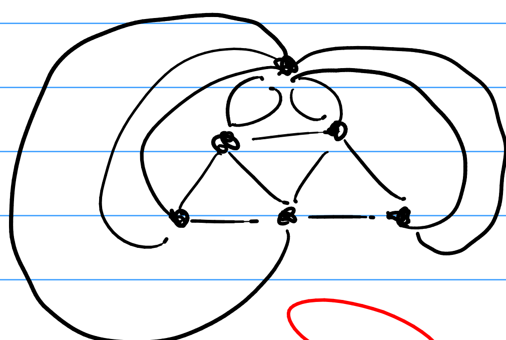
→ by thm no soln.



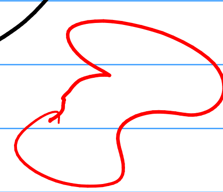
Cut puzzle



has Euler circuit.



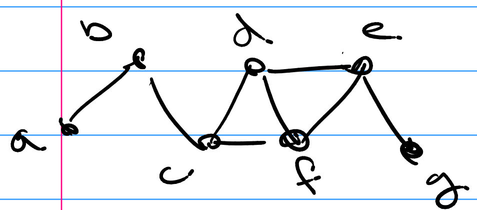
← has no soln.



Hamilton

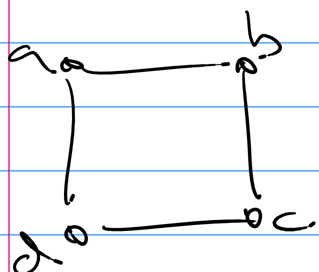
$G = (V, E)$

Hamilton Path: Simple path that passes through every vertex of G exactly once.



ex: a, b, c, d, f, e, g

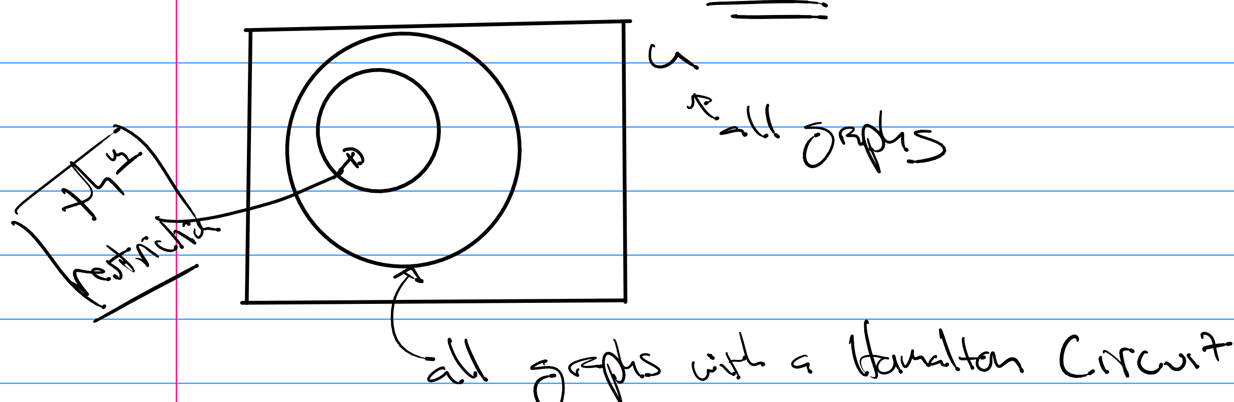
Hamilton Circuit: Hamilton path except only the first vertex = last vertex.



ex: a, b, c, d, a

For thm's → no iff thm

→ we do have suff.



Dirac's thm: $G = (V, E)$ is simple, $|V| = n \geq 3$

and $\forall v \in V \ \deg(v) \geq n/2$

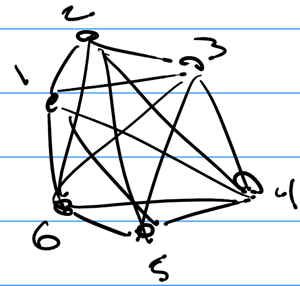
then G has a Hamilton Circuit.

ORE's $G = (V,E)$ is simple, $|V| = n \geq 3$

such that $\deg(v_i) + \deg(v_j) \geq n$ for all non-adj v_i, v_j

$\rightarrow$ G has a Hamilton Circuit.

ex K_6



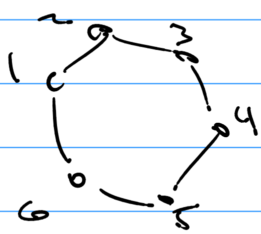
Dirac? check $\deg(v) \geq 6/2 = 3$

true. b/c $\deg(v) = 5$ always

ORE's? check $\deg(v_i) + \deg(v_j) \geq 6$

v_i, v_j are non-adj. (can't use this)

ex C_6



H.C. 1, 2, 3, 4, 5, 6, 1

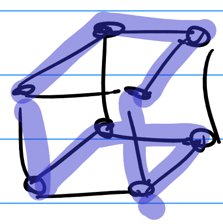
Dirac? $\deg(v) \geq 3$ False.

ORE's $\deg(v_1) + \deg(v_2) \geq 6$

False $2 + 2 = 4 \not\geq 6$

False.

ex Q_3



Dirac? $\deg(v) \geq 8/2 = 4$

False.

ORE's $\deg(v_i) + \deg(v_j) \geq 8$

$3 + 3 = 6 \not\geq 8$

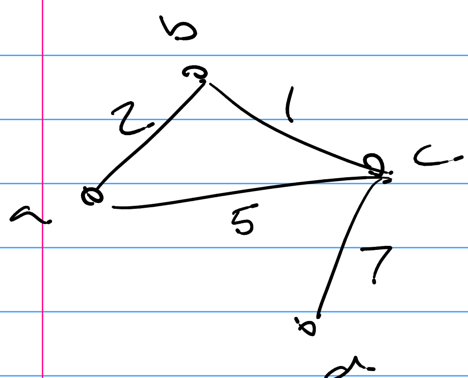
False.

Note: $K_3, K_4, K_5, \ldots$ all have Hamilton circuits

$$|\text{Hamilton Circuits}| = (n-1)(n-2)\cdots(1) = (n-1)!.$$

Apply to weighted graphs.

$G = (V, E)$ with weight function $w(e) =$ weight



path: seq of edges

length: sum of weights

(Note: if you let $w(e) = 1$
old length = our new idea of length)

a, b, c, a, b
2 1 5 2
$\Rightarrow$ length $= 10$

Q shortest path?

Q shortest Hamilton Circuit for K_n?